

DIAGONALIZING OPERATORS WITH REFLECTION SYMMETRY

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Dedicated to the memory of I.E. Segal

ABSTRACT. Let U be an operator in a Hilbert space \mathcal{H}_0 , and let $\mathcal{K} \subset \mathcal{H}_0$ be a closed and invariant subspace. Suppose there is a period-2 unitary operator J in \mathcal{H}_0 such that $JUJ = U^*$, and $PJP \geq 0$, where P denotes the projection of \mathcal{H}_0 onto \mathcal{K} . We show that there is then a Hilbert space $\mathcal{H}(\mathcal{K})$, a contractive operator $W: \mathcal{K} \rightarrow \mathcal{H}(\mathcal{K})$, and a selfadjoint operator $S = S(U)$ in $\mathcal{H}(\mathcal{K})$ such that $W^*W = PJP$, W has dense range, and $SW = WUP$. Moreover, given (\mathcal{K}, J) with the stated properties, the system $(\mathcal{H}(\mathcal{K}), W, S)$ is unique up to unitary equivalence, and subject to the three conditions in the conclusion. We also provide an operator-theoretic model of this structure where $U|_{\mathcal{K}}$ is a pure shift of infinite multiplicity, and where we show that $\ker(W) = 0$. For that case, we describe the spectrum of the selfadjoint operator $S(U)$ in terms of structural properties of U . In the model, U will be realized as a unitary scaling operator of the form

$$f(x) \mapsto f(cx), \quad c > 1,$$

and the spectrum of $S(U_c)$ is then computed in terms of the given number c .

1. INTRODUCTION

The paper is motivated by two problems one from mathematical physics, and the other from the interface of integral transforms and interpolation theory. The first problem is that of changing the spectrum of an operator, or a one-parameter group of operators, with a view to getting a new spectrum with physical desiderata (see, e.g., [Seg98]), for example creating a mass gap, and still preserving quasi-equivalence of the two underlying operator systems. In the other problem we study how Hilbert space functional completions change under the variation of a parameter in the integral kernel of the transform in question. The motivating example here is derived from a certain version of the Segal–Bargmann transform. For more detail on the background and the applications alluded to in the Introduction, we refer to the two previous joint papers [JoO198] and [JoO199], as well as [Nee94] and [Hal98].

Let U be an operator in a Hilbert space \mathcal{H}_0 , and let J be a period-2 unitary operator in \mathcal{H}_0 such that

$$(1.1) \quad JUJ = U^*.$$

We think of (1.1) as a reflection symmetry for the given operator U . In this case, U and its adjoint U^* have the same spectrum, but, of course, U need not be selfadjoint.

1991 *Mathematics Subject Classification.* 47A05, 47A66, 47B15.

Key words and phrases. Operators in Hilbert space, reflection, reproducing kernel Hilbert space, Knapp-Stein operator, singular integrals.

Work supported in part by the National Science Foundation.

Nonetheless, we shall think of (1.1) as a notion which generalizes selfadjointness. As an example, let the Hilbert space $\mathcal{H}_0 = L^2(\mathbb{T})$,

$$(1.2) \quad (Uf)(z) = zf(z), \quad f \in L^2(\mathbb{T}), \quad z \in \mathbb{T},$$

and

$$(1.3) \quad Jf(z) = f(\bar{z}).$$

The space $L^2(\mathbb{T})$ is from Haar measure on the circle group $\mathbb{T} = \{z \in \mathbb{C} ; |z| = 1\}$. It clear that (1.1) then holds. If $\mathcal{K} = H^2(\mathbb{T})$ is the Hardy space of functions, $f(z) = \sum_{n=0}^{\infty} c_n z^n$, with $\|f\|^2 = \sum_{n=0}^{\infty} |c_n|^2 < \infty$, then we also have

$$(1.4) \quad PJP \geq 0$$

where P denotes the projection onto $H^2(\mathbb{T})$. In fact

$$(1.5) \quad \langle f, Jf \rangle = |c_0|^2,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\mathbb{T})$. While our result applies to the multiplicity-one shift, this is a degenerate situation, and the nontrivial applications are for the case of infinite multiplicity.

There is in fact an infinite-multiplicity version of the above which we proceed to describe. Let $0 < s < 1$ be given, and let \mathcal{H}_s be the Hilbert space whose norm $\|f\|_s$ is given by

$$(1.6) \quad \|f\|_s^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{f(x)} |x - y|^{s-1} f(y) dx dy.$$

Let $a \in \mathbb{R}_+$ be given, and set

$$(1.7) \quad (U(a)f)(x) = a^{s+1} f(a^2 x).$$

It is clear that then $a \mapsto U(a)$ is a unitary representation of the multiplicative group \mathbb{R}_+ acting on the Hilbert space \mathcal{H}_s . It can be checked that $\|f\|_s$ in (1.6) is finite for all $f \in C_c(\mathbb{R})$ (= the space of compactly supported functions on the line). Now let $\mathcal{K} (= \mathcal{K}_s)$ be the closure of $C_c(-1, 1)$ in \mathcal{H}_s relative to the norm $\|\cdot\|_s$ of (1.6). It is then immediate that $U(a)$, for $a > 1$, leaves \mathcal{K}_s invariant, i.e., it restricts to a semigroup of isometries $\{U(a) ; a > 1\}$ acting on \mathcal{K}_s . Setting

$$(1.8) \quad (Jf)(x) = |x|^{-s-1} f\left(\frac{1}{x}\right), \quad x \in \mathbb{R} \setminus \{0\},$$

we check that J is then a period-2 unitary in \mathcal{H}_s , and that

$$(1.9) \quad JU(a)J = U(a)^* = U(a^{-1})$$

and

$$(1.10) \quad \langle f, Jf \rangle_{\mathcal{H}_s} \geq 0, \quad \forall f \in \mathcal{K}_s,$$

where $\langle \cdot, \cdot \rangle_{\mathcal{H}_s}$ is the inner product

$$(1.11) \quad \langle f_1, f_2 \rangle_{\mathcal{H}_s} := \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{f_1(x)} |x - y|^{s-1} f_2(y) dx dy.$$

In fact, if $f \in C_c(-1, 1)$, the expression in (1.10) works out as the following reproducing kernel integral:

$$(1.12) \quad \int_{-1}^1 \int_{-1}^1 \overline{f(x)} (1 - xy)^{s-1} f(y) dx dy,$$

and we refer to [JoOl98, JoOl99] for more details on this example.

As an application of our result, we will show that, if $a > 1$, then $U(a)|_{\mathcal{K}_s}$ induces a selfadjoint operator $S(a)$ in a Hilbert space $\mathcal{H}(\mathcal{K}_s)$, and there is a contraction $W: \mathcal{K}_s \rightarrow \mathcal{H}(\mathcal{K}_s)$, with

$$(1.13) \quad \ker(W) = 0,$$

such that

$$(1.14) \quad W^*W = PJP,$$

$$(1.15) \quad S(a)W = WU(a)P,$$

and

$$(1.16) \quad \text{spectrum}(S(a)) = \{a^{s-1-2n}; n = 0, 1, 2, \dots\}.$$

What is important in this application is the property (1.13). So the properties in this case for W are $\|W\| \leq 1$, $\ker(W^*) = \ker(W) = 0$. While of course $U(a)|_{\mathcal{K}_s}$ and $S(a)$ cannot be unitarily equivalent, then W nonetheless defines a strong notion of equivalence (quasi-equivalence) for the two semigroups $U(a)|_{\mathcal{K}_s}$ and $S(a)$, $a > 1$, specified by the intertwining property

$$(1.17) \quad S(a)W = WU(a)P.$$

In particular, since both W and W^* have dense range in the respective Hilbert spaces \mathcal{K} and $\mathcal{H}(\mathcal{K})$, it follows that the partial isometry part L in the polar decomposition $W = L(W^*W)^{1/2} = L(PJP)^{1/2}$, is in fact a *unitary* isomorphism of \mathcal{K} onto $\mathcal{H}(\mathcal{K})$. The intertwining property for W^*W of the polar decomposition is

$$(1.18) \quad (W^*W)UP = PU^*(W^*W).$$

But this cannot be iterated, so there is *not* an analogous relation for the factors $(W^*W)^{1/2}$ and L . The properties of W and S in this example imply that UP is in fact a pure shift (i.e., the unitary part of the isometry $U|_{\mathcal{K}_s}$ of the Wold decomposition is trivial, and moreover the backwards shift PU^* has a cyclic vector. The second conclusion is unique to this example, and follows from the fact that $S = S(a)$ has simple spectrum.

Proposition 1.1. *The isometry UP is a pure shift.*

Proof. The result may be read off from the following estimate:

$$(1.19) \quad \|PU^{*k}W^*\psi\| = \|W^*S(a^k)\psi\| \leq \|S(a^k)\psi\| \leq a^{k(s-1)}\|\psi\| \xrightarrow{k \rightarrow \infty} 0,$$

the estimate being valid for all $\psi \in \mathcal{H}(\mathcal{K})$. Since $\ker(W) = 0$, $W^*\mathcal{H}(\mathcal{K})$ is dense in \mathcal{K} , so we have $\lim_{k \rightarrow \infty} \|PU^{*k}\varphi\| = 0$ for all $\varphi \in \mathcal{K}$, and this last property is equivalent to $U|_{\mathcal{K}_s}$ being a pure shift on \mathcal{K}_s . \square

The restriction on s remains $0 < s < 1$. It follows in fact from [JoOl98, JoOl99] that the multiplicity of this shift is ∞ , i.e., that if $a > 1$, the dimension of $\mathcal{K}_s \ominus U(a)\mathcal{K}_s$ is infinite.

The simplest case of a system (\mathcal{H}_0, J) with J as a reflection is that of $\mathcal{H}_0 = \mathcal{H} \oplus \mathcal{H}$ and $J = I \oplus (-I)$, i.e.,

$$(1.20) \quad J(h_1 \oplus h_2) = h_1 \oplus (-h_2), \quad h_1, h_2 \in \mathcal{H}.$$

In many applications of this, it will further be given that \mathcal{H} is a reproducing kernel Hilbert space in the sense of [Aro50]. Suppose this is the case, and that $Q(\cdot, \cdot)$ is

the corresponding reproducing kernel. We then have \mathcal{H} realized as a Hilbert space of \mathbb{C} -valued functions $h(\cdot)$ defined on some set Ω , and Q is a function on $\Omega \times \Omega$ such that $Q(z, \cdot) \in \mathcal{H}$ for all $z \in \Omega$, and

$$(1.21) \quad \langle Q(z, \cdot), h \rangle = h(z) \quad \text{for all } h \in \mathcal{H}.$$

In this case, we will use Q in identifying a class of subspaces $\mathcal{K} \subset \mathcal{H} \oplus \mathcal{H}$ such that

$$(1.22) \quad \langle k, Jk \rangle \geq 0 \quad \text{for all } k \in \mathcal{K}.$$

We now describe such a class of spaces \mathcal{K} . Let $D := \{z \in \mathbb{C} ; |z| < 1\}$. It will be stated in an abstract setting, and the applications to interpolation theory will be given in Section 6 below.

Proposition 1.2. *Let \mathcal{H} be a reproducing kernel Hilbert space corresponding to a kernel function*

$$(1.23) \quad Q: \Omega \times \Omega \longrightarrow \mathbb{C},$$

and let $\Omega_0 \subset \Omega$ be a subset. Let a function

$$(1.24) \quad \varphi: \Omega_0 \longrightarrow \bar{D}$$

be given, and let $\mathcal{K}_\varphi \subset \mathcal{H} \oplus \mathcal{H}$ be defined as the closed span of

$$(1.25) \quad \left\{ \begin{pmatrix} Q(z, \cdot) \\ \varphi(z) Q(z, \cdot) \end{pmatrix} ; z \in \Omega_0 \right\} \subset \begin{pmatrix} \mathcal{H} \\ \mathcal{H} \end{pmatrix}^\oplus.$$

(i) *Then (1.22) holds for $J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ if and only if*

$$(z_1, z_2) \longmapsto \left(1 - \overline{\varphi(z_1)}\varphi(z_2)\right) Q(z_1, z_2)$$

is positive definite on Ω_0 .

(ii) *If instead $\varphi: \Omega_0 \rightarrow \mathbb{C}$, and $J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, then (1.22) holds if and only if*

$$(z_1, z_2) \longmapsto \left(\overline{\varphi(z_1)} + \varphi(z_2)\right) Q(z_1, z_2)$$

is positive definite on Ω_0 .

Proof. The result follows from a substitution of the vectors in (1.25) into the positivity requirement (1.22), and computing out the answer for the two cases of reflection J , i.e., $J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ and $J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. We refer to Section 6 for more details, and additional comments on applications to interpolation theory. \square

2. PURE ISOMETRIES

It is well known that pure isometries (alias shifts) of infinite multiplicity play a role in the harmonic analysis of wavelets, see [BrJo97b], and in the Lax–Phillips version of scattering theory for the wave equation [LaPh89]. Let V be a shift in a Hilbert space \mathcal{K} , and let

$$(2.1) \quad \mathcal{L} := \mathcal{K} \ominus V\mathcal{K};$$

then

$$(2.2) \quad \mathcal{K} = \sum_{n=0}^{\infty} \oplus V^n \mathcal{L}$$

as a direct sum. But for every nonzero $l \in \mathcal{L}$, and $z \in D := \{z \in \mathbb{C} ; |z| < 1\}$, the vector

$$(2.3) \quad f = f(l, z) := l \oplus zVl \oplus z^2V^2l \oplus \dots$$

is an eigenvector of V^* , i.e.,

$$(2.4) \quad V^*f = zf,$$

and $\|f\|^2 = (1 - |z|^2)^{-1} \|l\|^2$. In fact, as l varies over $\mathcal{L} \setminus \{0\}$, the vectors

$$(2.5) \quad \{f(l, z^n) ; n = 1, 2, \dots\}$$

span a dense subspace in \mathcal{K} . This is true for every $z \in D$ fixed; so it is clear from this that there is a variety of ways of creating selfadjoint, and normal, realizations of a given V , i.e., solutions to the problem

$$(2.6) \quad WV = NW.$$

Specifically, there is a Hilbert space $\mathcal{H}(\mathcal{K})$, a bounded operator $W : \mathcal{K} \rightarrow \mathcal{H}(\mathcal{K})$, and a normal operator N in $\mathcal{H}(\mathcal{K})$ such that (2.6) holds. This problem has been studied recently by Feldman [Fel99], and Agler et al. [AgMc98], but it is a different focus from ours. The reflection J plays a crucial role in our approach. It also makes our setting considerably more restrictive and it allows us to get solutions to the diagonalization problem which are *unique up to unitary equivalence*. More importantly, it gives an answer to a reflection problem from mathematical physics which we proceed to describe.

The approach (2.4) for V^* works for a wider class of operators than the backwards shift, namely the operators in the Cowen–Douglas classes, see [CoDo78], but we have not yet checked which of the Cowen–Douglas operators admit reflection symmetry.

Our next result will be stated for general bounded operators U which have reflection symmetry, and the symmetry is given in terms of a period-2 unitary J and a subspace \mathcal{K} which is invariant under U . From this we will then arrive at a selfadjoint realization S of U , and when (\mathcal{K}, J) is given, we will show that S is determined uniquely up to unitary equivalence. The result is interesting even if U is given at the outset to be unitary. In fact in an application from quantum field theory, U will be rather a unitary one-parameter group $\{U(t)\}_{t \in \mathbb{R}}$ of operators acting on a Hilbert space \mathcal{H}_0 , and \mathcal{K} will be a subspace in \mathcal{H}_0 which is invariant under $U(t)$ for $t \geq 0$. By Stone’s theorem [Var85], there is a selfadjoint Hamiltonian operator H (generally unbounded) in \mathcal{H}_0 such that

$$(2.7) \quad U(t) = e^{-itH}, \quad t \in \mathbb{R}.$$

In this application, we will have

$$(2.8) \quad JU(t)J = U(-t), \quad t \in \mathbb{R},$$

and J is referred to as “time-reversal” or “time-reflection”. The initial Hamiltonian might not have the right “physical” spectrum; for example, the spectrum of H might be all of \mathbb{R} , and what is desired would be a spectrum which is contained in \mathbb{R}_+ with a positive gap between 0 and the bottom of the “physical” spectrum. We will show that this can be achieved; in fact we will describe a selfadjoint realization $S = S(U)$ in the form of a semigroup

$$(2.9) \quad S(t) = e^{-t\hat{H}}$$

where \hat{H} is a selfadjoint operator in the new Hilbert space $\mathcal{H}(\mathcal{K})$, and the spectrum of \hat{H} will be “physical” in that it will be positive and there will be a “mass gap”, i.e., a positive gap between 0 and the lower bound for spectrum (\hat{H}) . But the key to passing from H to \hat{H} will be the given (\mathcal{K}, J) when $\mathcal{K} \subset \mathcal{H}_0$ is assumed invariant under $U(t)$, $t \geq 0$, and J is a time-reflection, i.e., J and $\{U(t)\}$ will satisfy (2.8). As we noted, the construction $H \rightsquigarrow \hat{H}$ with \hat{H} having a mass-gap will show, after the fact, that the initial semigroup of isometries $U(t)|_{\mathcal{K}}$, $t \geq 0$, will necessarily be a pure shift (and of infinite multiplicity). By this we mean that there is a unitary isomorphism between \mathcal{H}_0 and $L^2(\mathbb{R}, \mathcal{M})$ for some infinite-dimensional Hilbert space \mathcal{M} which intertwines $\{U(t)\}_{t \in \mathbb{R}}$ with translation on $L^2(\mathbb{R}, \mathcal{M})$. Specifically, there is a unitary isomorphism

$$(2.10) \quad Y: \mathcal{H}_0 \longrightarrow L^2(\mathbb{R}, \mathcal{M}), \text{ onto,}$$

such that

$$(2.11) \quad YU(t)Y^{-1}f(x) = f(x-t), \quad f \in L^2(\mathbb{R}, \mathcal{M}), \quad t \in \mathbb{R},$$

with the further property that

$$(2.12) \quad Y(\mathcal{K}) = L^2(\mathbb{R}_+, \mathcal{M}),$$

i.e., the functions in $L^2(\mathbb{R}, \mathcal{M})$ which are supported in the positive half line.

3. REFLECTION SYMMETRY

The following result provides the axiomatic setup for reflection symmetry in the form described above. With the given symmetry axioms, it provides the step $U \mapsto S(U)$ from a general operator U with symmetry to its selfadjoint version $S(U)$, and we show that $S(U)$ is unique up to unitary equivalence. The data that emerges is $(\mathcal{H}(\mathcal{K}), W, S)$, where

$$(3.1) \quad SW = WUP.$$

Here P denotes the projection onto the subspace \mathcal{K} which both is invariant for U and satisfies reflection positivity relative to the period-2 unitary J (i.e., the reflection). But in the general setting, the axioms allow $W: \mathcal{K} \rightarrow \mathcal{H}(\mathcal{K})$ to have nonzero kernel, and this represents some degree of non-uniqueness: for example, W may be a “small” (rank-one, say) projection, and S might be zero. Hence we shall focus on the setting when $\ker(W) = 0$, and we will say then that the two operators $U|_{\mathcal{K}}$ and S are *quasi-equivalent*. While the intertwining operator W is 1–1 with dense range, its inverse W^{-1} will be unbounded.

Theorem 3.1. *Let U be a bounded operator in a Hilbert space \mathcal{H}_0 . Let $\mathcal{K} \subset \mathcal{H}_0$ be an invariant subspace, and let P denote the projection of \mathcal{H}_0 onto \mathcal{K} . Let J be a period-2 unitary operator in \mathcal{H}_0 which satisfies*

$$(i) \quad JUJ = U^*$$

and

$$(ii) \quad PJP \geq 0.$$

(a) *Then there is a Hilbert space $\mathcal{H}(\mathcal{K})$ and a contractive operator*

$$W: \mathcal{K} \longrightarrow \mathcal{H}(\mathcal{K})$$

with dense range, and a bounded selfadjoint operator $S = S(U)$ in $\mathcal{H}(\mathcal{K})$ such that

$$(iii) \quad SW = WUP,$$

$$(iv) \quad W^*W = PJP,$$

and

$$(v) \quad \|S(U)\| \leq (\text{sp}(U^2))^{\frac{1}{2}},$$

where $\text{sp}(U^2)$ denotes the spectral radius of U^2 .

- (b) Given (i)–(ii) the data $(\mathcal{H}(\mathcal{K}), W, S)$ is unique up to unitary equivalence subject to the axioms (iii)–(iv). Specifically, suppose $(\mathcal{H}_i(\mathcal{K}), W_i, S_i)$, $i = 1, 2$, are two systems which both solve the extension problem, i.e., are extensions satisfying (iii)–(iv). Then there is a unitary isomorphism $T: \mathcal{H}_1(\mathcal{K}) \rightarrow \mathcal{H}_2(\mathcal{K})$ of $\mathcal{H}_1(\mathcal{K})$ onto $\mathcal{H}_2(\mathcal{K})$ which satisfies

$$(vi) \quad TW_1 = W_2$$

and

$$(vii) \quad TS_1 = S_2T.$$

- (c) There are operators U , with reflection symmetry, such that W from $(\mathcal{H}(\mathcal{K}), W, S)$ has

$$(viii) \quad \ker(W) = 0.$$

Proof. The proof is rather long and will be broken up into its three parts (a), (b), and (c). Part (a) asserts the existence of a selfadjoint realization of the given operator U , while part (b) is uniqueness up to unitary equivalence. Part (c) is an explicit construction which takes place in a certain reproducing kernel Hilbert space.

The following observation gives a more concrete understanding of axiom (ii) in part (a) of Theorem 3.1. Let J be a period-2 unitary operator in a Hilbert space \mathcal{H}_0 , and let \mathcal{H}_\pm be the respective eigenspaces corresponding to eigenvalues ± 1 of J . If P_+ is the projection onto \mathcal{H}_+ , then $J = 2P_+ - I$.

Lemma 3.2. *A closed subspace $\mathcal{K} \subset \mathcal{H}_0$ satisfies (ii) if and only if \mathcal{K} is the graph of a contractive operator Λ from \mathcal{H}_+ to \mathcal{H}_- . By this we mean that Λ is defined on a closed subspace $\mathcal{P} \subset \mathcal{H}_+$ and Λ maps \mathcal{P} contractively into \mathcal{H}_- . Hence $\mathcal{K} \simeq \{(p, \Lambda p) ; p \in \mathcal{P}\}$, or we will write simply $\mathcal{K} = G(\Lambda)$ and $\mathcal{P} = D(\Lambda)$ where G and D are used for graph and domain, respectively.*

Proof. The main idea in the proof is in [Phil], but we include a sketch. This will also give us a chance for introducing some terminology which will be needed later anyway. Suppose $\mathcal{K} \subset \mathcal{H}_0$ is a closed subspace which satisfies (ii). For $k \in \mathcal{K}$ we have $k = P_+k + P_-k$, where $P_- := I - P_+$ and $J = P_+ - P_-$. But $\langle k, Jk \rangle = \|P_+k\|^2 - \|P_-k\|^2$ for all $k \in \mathcal{K}$ by (ii), and if we define $\Lambda P_+k := P_-k$, then Λ is well-defined and contractive from $\mathcal{P} = P_+\mathcal{K}$ to $P_-\mathcal{K}$. The reasoning shows that the converse argument is also valid, so the lemma follows except for the assertion that $\mathcal{P} := P_+\mathcal{K}$ must be automatically closed. Let k_n be a sequence of vectors in \mathcal{K} such that $P_+k_n \rightarrow h_+ \in \mathcal{H}_+$. Then by (ii),

$$\|P_-(k_n - k_m)\| \leq \|P_+(k_n - k_m)\| \longrightarrow 0 \quad \text{as } n, m \longrightarrow \infty.$$

So the limit $\lim_{n \rightarrow \infty} P_- k_n = h_-$ exists in \mathcal{H}_- , and

$$k_n = P_+ k_n + P_- k_n \longrightarrow h_+ + h_-.$$

Since \mathcal{K} is assumed closed in \mathcal{H}_0 , we get $h_+ + h_- \in \mathcal{K}$, and $h_+ = P_+(h_+ + h_-) = \lim_{n \rightarrow \infty} P_+ k_n$. This shows that $P_+ \mathcal{K}$ is closed, and the proof is completed. \square

Proof of Theorem 3.1 continued. (a) Let the operator U be given as in the statement of the theorem. Let $\mathcal{K} \subset \mathcal{H}_0$ be the invariant subspace with projection P , and let J be the reflection. It is assumed to satisfy (i)–(ii). In view of (ii), we have

$$(3.2) \quad \langle k, Jk \rangle \geq 0 \quad \text{for all } k \in \mathcal{K},$$

where $\langle \cdot, \cdot \rangle$ denotes the given inner product from \mathcal{H}_0 . (Note that \mathcal{K} is not invariant under J , so the vector Jk is typically not in \mathcal{K} if k is.) Applying the Cauchy–Schwarz inequality, we get

$$(3.3) \quad |\langle k_1, Jk_2 \rangle|^2 \leq \langle k_1, Jk_1 \rangle \langle k_2, Jk_2 \rangle \quad \text{for all } k_1, k_2 \in \mathcal{K}.$$

The idea is to get a new Hilbert space $\mathcal{H}(\mathcal{K})$ from the form $\langle k_1, Jk_2 \rangle$, i.e., that this form should be the new inner product. So we must form the quotient space \mathcal{K}/\mathcal{N} where

$$(3.4) \quad \mathcal{N} = \{k \in \mathcal{K} ; \langle k, Jk \rangle = 0\}.$$

In view of (3.3), we get

$$(3.5) \quad \mathcal{N} = \{k_0 \in \mathcal{K} ; \langle k_0, Jk \rangle = 0 \text{ for all } k \in \mathcal{K}\}.$$

Since

$$(3.6) \quad \langle k_1, JUk_2 \rangle = \langle k_1, U^* Jk_2 \rangle = \langle Uk_1, Jk_2 \rangle \quad \text{for all } k_1, k_2 \in \mathcal{K},$$

we conclude that U passes to the quotient \mathcal{K}/\mathcal{N} and defines there a symmetric operator. When \mathcal{K}/\mathcal{N} is completed in the new norm $\|\cdot\|_J$,

$$(3.7) \quad \|k\|_J^2 := \langle k, Jk \rangle,$$

the induced operator becomes selfadjoint in this Hilbert space

$$(3.8) \quad \mathcal{H}(\mathcal{K}) := (\mathcal{K}/\mathcal{N})^\sim.$$

The induced operator will be denoted $S = S(U)$, and we will now show that it satisfies conditions (iii)–(v), starting with (v), i.e., showing first that $S(U)$ is a bounded operator in the Hilbert space $\mathcal{H}(\mathcal{K})$. The argument for boundedness is essentially in [JoOl98], but we include it here for the convenience of the reader.

Let $k \in \mathcal{K}$, and use recursion on (3.3) as follows:

$$\begin{aligned}
\|Uk\|_J^2 &= \langle Uk, JUk \rangle = \langle Uk, U^* Jk \rangle = \langle U^2 k, Jk \rangle \\
&\leq \langle U^2 k, JU^2 k \rangle^{\frac{1}{2}} \langle k, Jk \rangle^{\frac{1}{2}} \\
&\leq \langle U^4 k, JU^4 k \rangle^{\frac{1}{4}} \langle k, Jk \rangle^{\frac{1}{2} + \frac{1}{4}} \\
&\leq \dots \\
&\leq \left\langle U^{2^n} k, JU^{2^n} k \right\rangle^{\frac{1}{2^n}} \cdot \langle k, Jk \rangle^{\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n}} \\
&\leq \left\langle U^{2^{n+1}} k, Jk \right\rangle^{\frac{1}{2^n}} \cdot \|k\|_J^2 \\
&\leq \left\| U^{2^{n+1}} \right\|^{\frac{1}{2^n}} \cdot \|k\|^{\frac{1}{2^{n-1}}} \cdot \|k\|_J^2.
\end{aligned}$$

We have $\lim_{n \rightarrow \infty} \left\| U^{2^{n+1}} \right\|^{\frac{1}{2^n}} = \text{sp}(U^2) = \text{the spectral radius}$, and $\lim_{n \rightarrow \infty} \|k\|^{\frac{1}{2^{n-1}}} = 1$ if $k \neq 0$. We have therefore proved the estimate

$$\|Uk\|_J \leq (\text{sp}(U^2))^{\frac{1}{2}} \|k\|_J$$

for $k \in \mathcal{K}$, and it follows that the induced operator $S = S(U)$ on $\mathcal{H}(\mathcal{K}) = (\mathcal{K}/\mathcal{N})^\sim$ satisfies (v), as claimed. Since we already showed that S is selfadjoint, we conclude that S has bounded spectrum inside the interval

$$(3.9) \quad \left[-(\text{sp}(U^2))^{\frac{1}{2}}, (\text{sp}(U^2))^{\frac{1}{2}} \right] \subset \mathbb{R}.$$

If U on \mathcal{H}_0 is unitary, this is the interval $[-1, 1]$. If $U = U(t)$, $t \in \mathbb{R}$, is a group of operators, then $S = S(t)$, $t \geq 0$, is a semigroup of selfadjoint operators, and so

$$(3.10) \quad S(t) = S\left(\frac{t}{2}\right)^2 \geq 0$$

for all $t \geq 0$, and the spectrum of $S(t)$ is therefore positive in that case, and we get the representation

$$(3.11) \quad S(t) = e^{-t\hat{H}}, \quad t \geq 0,$$

for some (generally unbounded) selfadjoint operator \hat{H} in $\mathcal{H}(\mathcal{K})$. □

Proof of part (b). For a given operator U which has a pair (\mathcal{K}, J) defining a reflection symmetry, we showed in (a) that there is a system $(\mathcal{H}(\mathcal{K}), W, S)$ with a selfadjoint operator S in $\mathcal{H}(\mathcal{K})$, and an intertwining operator W , which satisfy (iii)–(v) in the statement of the theorem. We now prove that this system is unique up to unitary equivalence. So suppose there are two systems $(\mathcal{H}_i(\mathcal{K}), W_i, S_i)$, $i = 1, 2$, both satisfying (iii)–(iv) and with the two “extension” operators S_1 and S_2 both selfadjoint and bounded. We will now show that there is then a unitary isomorphism $T: \mathcal{H}_1(\mathcal{K}) \rightarrow \mathcal{H}_2(\mathcal{K})$ which defines the equivalence, i.e., which satisfies (vi) and (vii) in the theorem. We will make (vi) into a definition, setting

$$(3.12) \quad TW_1 k = W_2 k,$$

for $k \in \mathcal{K}$. Since both W_1 and W_2 satisfy (iv), we conclude that

$$\|W_1 k\|_J = 0 \iff k \in \mathcal{N} \iff \|W_2 k\|_J = 0,$$

or, stated equivalently,

$$\ker(W_i) = \mathcal{N} \quad \text{for } i = 1, 2,$$

where \mathcal{N} is defined in (3.4). Hence, formula (3.12) makes a good definition of a linear operator T mapping a dense subspace in $\mathcal{H}_1(\mathcal{K})$ into one in $\mathcal{H}_2(\mathcal{K})$. But property (iv) for W_1 and W_2 implies that T is also isometric, indeed

$$\|TW_1k\|_J^2 = \|W_2k\|_J^2 = \langle k, Jk \rangle = \|W_1k\|_J^2.$$

Hence T is a unitary isomorphism of $\mathcal{H}_1(\mathcal{K})$ onto $\mathcal{H}_2(\mathcal{K})$. Using now (iii) for the two systems, we get

$$(TS_1)W_1k = TW_1Uk = W_2Uk = S_2W_2k = (S_2T)W_1k \quad \text{for all } k \in \mathcal{K}.$$

Since W_1 has dense range, we get the desired intertwining property (vii) as claimed in the theorem. \square

Proof of part (c). The assertion in part (c) is that there are examples where the induction $U \rightsquigarrow S(U)$ has intertwining operator W with zero kernel, or equivalently, $\mathcal{N} = \{0\}$. We already mentioned this in (1.6)–(1.8) of Section 1, and in fact this is a one-parameter semigroup of isometries $U(a)P_{\mathcal{K}_s}$, $a > 1$. In fact, it arises as the restriction to an invariant subspace of a unitary one-parameter group. It is a representation $U(a)$, $a \in \mathbb{R}_+$, of the multiplicative group \mathbb{R}_+ , or equivalently, via $a = e^t$, a representation of the additive group \mathbb{R} . We get as a corollary of (c) that $\{U_s(e^t)\}_{t \in \mathbb{R}}$ is equivalent to the group of translations on $L^2(\mathbb{R}, \mathcal{M})$ for some infinite-dimensional Hilbert space \mathcal{M} as described in (2.10)–(2.12) in the conclusion of Section 2 above.

Now recall the Hilbert space \mathcal{H}_s and its subspace \mathcal{K}_s from Section 1. When $0 < s < 1$, \mathcal{H}_s is defined by the norm $\|\cdot\|_s$ from (1.6) and the subspace \mathcal{K}_s is the completion of $C_c(-1, 1)$ in the $\|\cdot\|_s$ -norm. We may pick some $a > 1$, and consider the isometry $U_s(a)|_{\mathcal{K}_s}$ of \mathcal{K}_s . From (1.8) we see that J also depends on s . The new inner product is

$$(3.13) \quad \langle k_1, k_2 \rangle_J := \langle k_1, Jk_2 \rangle_{\mathcal{H}_s}$$

(defined for $k_1, k_2 \in \mathcal{K}_s$), and depends on s as well. It is worked out explicitly in (1.12). It follows from (1.11) that $\langle \cdot, \cdot \rangle_{\mathcal{H}_s}$ is defined from the integral kernel $|x - y|^{s-1}$. The corresponding operator A_s is a special case of the Knapp–Stein intertwining operator, see [KnSt80]. (See also [Sal62] and [Rad98].) This operator $A_s(n)$ is defined more generally and also in \mathbb{R}^n . Then the integral kernel is $|x - y|^{s-n}$, and $0 < s < n$. If Δ is the positive Laplace operator in \mathbb{R}^n , i.e., $\Delta = \sum_{j=1}^n \left(\frac{1}{i} \frac{\partial}{\partial x_j} \right)^2$, then it is shown in [Ste70, Lemma 2, p. 117] that $A_s = \Delta^{-\frac{s}{2}}$, and the Fourier transform of $|x|^{s-n}$ is

$$\left(\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) / \Gamma\left(\frac{n-s}{2}\right) \right) \cdot |\xi|^{-s}.$$

Hence up to a constant, the norm $\|\cdot\|_s$ of (1.11) may be rewritten as

$$(3.14) \quad \int_{\mathbb{R}} |\xi|^{-s} \left| \hat{f}(\xi) \right|^2 d\xi,$$

and the inner product $\langle \cdot, \cdot \rangle_s$ as

$$(3.15) \quad \int_{\mathbb{R}} |\xi|^{-s} \overline{\hat{f}_1(\xi)} \hat{f}_2(\xi) d\xi,$$

where

$$(3.16) \quad \hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x) dx$$

is the usual Fourier transform suitably extended to \mathcal{H}_s , using Stein's singular integrals. Intuitively, \mathcal{H}_s consists of functions on \mathbb{R} which arise as $\left(\frac{d}{dx}\right)^s f_s$ for some f_s in $L^2(\mathbb{R})$. This also introduces a degree of “non-locality” into the theory, and the functions in \mathcal{H}_s cannot be viewed as locally integrable, although \mathcal{H}_s for each s , $0 < s < 1$, contains $C_c(\mathbb{R})$ as a dense subspace. In fact, formula (3.14), for the norm in \mathcal{H}_s , makes precise in which sense elements of \mathcal{H}_s are “fractional” derivatives of locally integrable functions on \mathbb{R} , and that there are elements of \mathcal{H}_s (and of \mathcal{K}_s) which are not locally integrable. On the other hand, vectors in \mathcal{H}_s are not too singular: for example the Dirac function δ is not in \mathcal{H}_s . To see this, pick some approximate identity $\varphi_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \delta$, say $\varphi \in C_c(-1, 1)$, $\varphi > 0$, $\int \varphi(x) dx = 1$, and set $\varphi_\varepsilon(x) = \frac{1}{\varepsilon} \varphi\left(\frac{x}{\varepsilon}\right)$; then a calculation shows that

$$(3.17) \quad \|\varphi_\varepsilon\|_{\mathcal{H}_s}^2 = C_s \varepsilon^{s-1}$$

for some positive constant C_s . Hence δ is not in \mathcal{H}_s , and then of course also not in the subspace \mathcal{K}_s .

Nonetheless, if we pass to the new norm $\|f\|_J^2 = \|f\|_{\mathcal{H}(\mathcal{K}_s)}^2 = \langle f, Jf \rangle_s$ of (3.13), then from (1.12) we get

$$(3.18) \quad \|\varphi_\varepsilon\|_J^2 = \mathcal{O}(\varepsilon^2).$$

Hence the limit $\varphi_\varepsilon \rightarrow \delta$ defines a bounded linear functional on $\mathcal{H}(\mathcal{K}_s)$, relative to the norm $\|\cdot\|_J$ on that Hilbert space. From the Riesz lemma, and the definition of $\mathcal{H}(\mathcal{K}_s)$, we conclude that δ is in $\mathcal{H}(\mathcal{K}_s)$. The same argument shows that the distributions $\delta^{(n)} := \left(\frac{d}{dx}\right)^n \delta$ given by

$$(3.19) \quad \delta^{(n)}(\phi) = (-1)^n \frac{d^n \phi}{dx^n}(0)$$

for $\phi \in C_c^\infty(-1, 1)$, are also in $\mathcal{H}(\mathcal{K}_s)$. In fact, the norm computes out as

$$(3.20) \quad \left\| \delta^{(n)} \right\|_J^2 = n! (1-s)(2-s) \cdots (n-s) \quad \text{for } n = 0, 1, 2, \dots$$

In the next lemma we provide the detailed proof of the fact that the iterated derivatives $\left(\frac{d}{dx}\right)^n \delta =: \delta^{(n)}$ of the Dirac delta function are all in the completion of $C_c^\infty(-1, 1)$ relative to the “new” norm of the Hilbert space $\mathcal{H}(\mathcal{K}_s)$. But recall that δ , or its derivatives, are not in \mathcal{K}_s .

Lemma 3.3. *For the Dirac mass and its derivatives, we have $\delta^{(n)} \in \mathcal{H}(\mathcal{K}_s)$, $n = 0, 1, 2, \dots$. The restriction on s is, as before, $0 < s < 1$.*

Proof. First note that if $\phi \in C_c^\infty(-1, 1)$, then

$$(3.21) \quad \int_{-1}^1 \phi(x) (1-xy)^{s-1} dx$$

restricts to a C^∞ -function on $[-1, 1]$. By this we mean that there is a C^∞ -function φ_s on \mathbb{R} such that

$$(3.22) \quad \varphi_s(y) = \int_{-1}^1 \phi(x) (1 - xy)^{s-1} dx$$

holds for all y in $[-1, 1]$. Hence, if F is a distribution with compact support in $[-1, 1]$, then

$$(3.23) \quad \langle \varphi_s, F \rangle = F(\varphi_s)$$

is well-defined. The same argument shows that $\langle (1 - \cdot y)^{s-1}, F \rangle$ is well-defined, and that

$$y \mapsto \langle (1 - \cdot y)^{s-1}, F \rangle$$

is also C^∞ up to the endpoints in the closed interval $I = [-1, 1]$. Hence, the distribution F may be applied again, and we get the expression

$$(3.24) \quad \|F\|_{\mathcal{H}(\mathcal{K}_s)}^2 := \int_I \int_I \overline{F(x)} (1 - xy)^{s-1} F(y) dx dy.$$

Moreover, if $\phi \in C_c^\infty(-1, 1)$, then

$$\langle W\phi, F \rangle_{\mathcal{H}(\mathcal{K}_s)} = \int_I \int_I \overline{\phi(x)} (1 - xy)^{s-1} F(y) dx dy$$

is well-defined in the distribution sense, and

$$\left| \langle W\phi, F \rangle_{\mathcal{H}(\mathcal{K}_s)} \right| \leq \|W\phi\|_{\mathcal{H}(\mathcal{K}_s)} \|F\|_{\mathcal{H}(\mathcal{K}_s)},$$

where $\|F\|_{\mathcal{H}(\mathcal{K}_s)}$ is the expression (3.24). Hence for each $n = 0, 1, 2, \dots$, we must show the following implication:

$$(3.25) \quad \langle W\phi, F \rangle_{\mathcal{H}(\mathcal{K}_s)} = 0 \text{ for all } \phi \in C_c^\infty(-1, 1) \implies \langle \delta^{(n)}, F \rangle_{\mathcal{H}(\mathcal{K}_s)} = 0.$$

The interpretation of the brackets $\langle \cdot, \cdot \rangle_{\mathcal{H}(\mathcal{K}_s)}$ is in the sense of distributions as noted. In particular,

$$(3.26) \quad \langle \delta^{(n)}, F \rangle_{\mathcal{H}(\mathcal{K}_s)} = (s-1) \cdots (s-n) \int_I y^n F(y) dy,$$

where $\int_I y^n F(y) dy$ is really the compactly supported distribution F evaluated at the monomial y^n . Recall, it is assumed that the distribution F is supported in I . Now pick $\phi \in C_c^\infty(-1, 1)$ such that $\phi > 0$, and $\int_I \phi(x) dx = 1$, and let $\phi_\varepsilon(x) = \frac{1}{\varepsilon} \phi\left(\frac{x}{\varepsilon}\right)$, for $0 < \varepsilon < 1$. We prove next that

$$(3.27) \quad \lim_{\varepsilon \rightarrow 0} \langle W\phi_\varepsilon^{(n)}, F \rangle_{\mathcal{H}(\mathcal{K}_s)} = \langle \delta^{(n)}, F \rangle_{\mathcal{H}(\mathcal{K}_s)},$$

where both sides are understood in the sense of distributions. But we also have $\langle W\phi_\varepsilon^{(n)}, F \rangle = 0$ for all $\varepsilon > 0$, by the assumption in (3.25). To complete the proof we will then only need to check that

$$(3.28) \quad \sup_{0 < \varepsilon < 1} \|W\phi_\varepsilon^{(n)}\|_{\mathcal{H}(\mathcal{K}_s)} < \infty.$$

Explicitly,

$$(3.29) \quad \|W\phi_\varepsilon^{(n)}\|_{\mathcal{H}(\mathcal{K}_s)}^2 = \int_I \int_I \phi_\varepsilon^{(n)}(x) (1 - xy)^{s-1} \phi_\varepsilon^{(n)}(y) dx dy,$$

and this last expression can be estimated directly: If $n \in \{0, 1, 2, \dots\}$, there is a constant C_n ($< \infty$) such that the $\int_I \int_I \dots dx dy$ term in (3.29) is estimated by C_n . In particular, we have the desired estimate (3.28). The left-hand side of (3.27) may therefore be estimated by $\|F\|_{\mathcal{H}(\mathcal{K}_s)} \cdot C_n$. Since $\langle W\phi_\varepsilon^{(n)}, F \rangle_{\mathcal{H}(\mathcal{K}_s)} = 0$ for all n and all ε , by assumption, see (3.25), we will then have $\langle \delta^{(n)}, F \rangle_{\mathcal{H}(\mathcal{K}_s)} = 0$, which is the claim.

It remains to check that the limit (as $\varepsilon \rightarrow 0$) in (3.27) is as stated. The argument is much as the previous one, so we will merely sketch the details for the case of $n = 0$: Since F is an distribution with support in $I = [-1, 1]$, we need to check that

$$(3.30) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_I \phi\left(\frac{x}{\varepsilon}\right) (1 - xy)^{s-1} dx = 1$$

and

$$(3.31) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left(\frac{d}{dy}\right)^m \int_I \phi\left(\frac{x}{\varepsilon}\right) (1 - xy)^{s-1} dx = 0$$

for all $m \in \mathbb{N}$; and both of these limits can be verified by calculus. Indeed the left-hand side in (3.30) is of the order

$$L_\varepsilon^{(s)}(y) := \begin{cases} \frac{(1 + \varepsilon y)^s - (1 - \varepsilon y)^s}{2\varepsilon s y} & \text{if } y \neq 0, \\ 1 & \text{if } y = 0, \end{cases}$$

which is differentiable in y , for every $\varepsilon \in \mathbb{R}_+$. The corresponding expression in (3.31) is $\mathcal{O}(\varepsilon^m)$, $m = 1, 2, \dots$. Since the distribution is of compact support (in I) we also have, for some $m \in \mathbb{N}$, the estimate

$$|F(\psi)| \leq \text{Const.} \cdot \max_{0 \leq k \leq m} \max_{x \in I} |\psi^{(k)}(x)|$$

for all $\psi \in C^\infty(\mathbb{R})$.

Applying this to the functions $\psi (= L_\varepsilon)$ in the left-hand side of (3.30), we finally arrive at the desired conclusion (3.27). This completes the proof of the lemma. \square

Hence if $f \in \mathcal{K}_s$, $Wf \in \mathcal{H}(\mathcal{K}_s)$, we get the inner product $\langle \delta^{(n)}, Wf \rangle_J$ is well-defined. A calculation yields

$$(3.32) \quad \langle \delta^{(n)}, Wf \rangle_J = (s-1)(s-2) \cdots (s-n) \int_{-1}^1 x^n f(x) dx.$$

However, if f is not locally integrable, then the right-hand side in (3.32) must be understood as a singular integral, see, e.g., [Ste70, Chapters V.1–2].

Recall that \mathcal{K}_s is obtained as the completion of $C_c(-1, 1)$ relative to the norm $\|\cdot\|_s$ of (1.11). If f is in $C_c(-1, 1)$, then the Fourier transform

$$(3.33) \quad \hat{f}(\xi) = \int_{-1}^1 f(x) e^{-ix\xi} dx$$

of (3.16) clearly has an entire analytic extension, i.e., it extends to complex values of ξ as an entire analytic function with exponential growth factor $e^{|\text{Im}\xi|}$, $\xi \in \mathbb{C}$. We wish to show that this also holds for $f \in \mathcal{N} \subset \mathcal{K}_s$. Note if $f \in \mathcal{N}$, it has finite

$\|\cdot\|_s$ -norm, and

$$(3.34) \quad \int_{-1}^1 \int_{-1}^1 \overline{f(x)} (1-xy)^{s-1} f(y) dx dy = 0,$$

or rather $\|f\|_J = 0$. Since f can be rather singular, the claim requires a proof. We have $Wf = 0$, and the Dirac measures δ_x , for $x \in \mathbb{R}$, $|x| < 1$, are in $\mathcal{H}(\mathcal{K}_s)$. Hence $\langle \delta_x, Wf \rangle_J = 0$. But a calculation yields, for $x \in (-1, 1) =: I$,

$$(3.35) \quad \langle \delta_x, Wf \rangle_J = \int_{-1}^1 (1-xy)^{s-1} f(y) dy.$$

Let $x \in I \setminus \{0\}$, and multiply by $|x|^{1-s}$, to get

$$\int_{-1}^1 \left| \frac{1}{x} - y \right|^{s-1} f(y) dy = 0,$$

and so $(A_s f)\left(\frac{1}{x}\right) = 0$. We conclude that Af is supported in the interval if f is in \mathcal{N} . This localizes the computation of

$$(3.36) \quad \|f\|_s^2 = \int_{\mathbb{R}} \overline{f(x)} A_s f(x) dx,$$

but still interpreted as a singular integral.

Since $\|f\|_s < \infty$, and $f \in \mathcal{K}_s$, there is a sequence $\varphi_n \in C_c^\infty(-1, 1)$ such that $\lim_{n \rightarrow \infty} \|f - \varphi_n\|_s = 0$. Then of course also

$$(3.37) \quad \lim_{n \rightarrow \infty} \|\varphi_n\|_s = \|f\|_s < \infty.$$

But

$$(3.38) \quad \|\varphi_n\|_s^2 = C_s \int_{\mathbb{R}} |\xi|^{-s} |\hat{\varphi}_n(\xi)|^2 d\xi$$

by (3.14). It follows that there is a subsequence φ_{n_i} such that $\hat{\varphi}_{n_i}(\cdot)$ converges pointwise almost everywhere on \mathbb{R} . We wish to use Montel's theorem [Hil62, v. II, Theorem 15.3.1] to conclude that the Fourier transform \hat{f} of f also has an entire analytic extension. To do this we need only check that $\hat{\varphi}_{n_i}(\zeta)$, $\zeta \in \mathbb{C}$, is an equicontinuous family. Now pick $\zeta_1, \zeta_2 \in \mathbb{C}$, and consider

$$\hat{\varphi}_{n_i}(\zeta_1) - \hat{\varphi}_{n_i}(\zeta_2) = \int_{-1}^1 \varphi_{n_i}(x) \{e^{-ix\zeta_1} - e^{-ix\zeta_2}\} dx.$$

Let $E(x) := e^{-ix\zeta_1} - e^{-ix\zeta_2}$, and pick $\psi \in C_c^\infty(\mathbb{R})$ such that $\psi \equiv 1$ on $\bar{I} = [-1, 1]$. Continuing the calculation, we get

$$\begin{aligned} \int_{-1}^1 \varphi_{n_i}(x) E(x) dx &= \int_{\mathbb{R}} \varphi_{n_i}(x) \psi(x) E(x) dx \\ &= \int_{\mathbb{R}} (\Delta^{-\frac{s}{2}} \varphi_{n_i}(x)) (\Delta^{\frac{s}{2}} \psi E(x)) dx \end{aligned}$$

and

$$\begin{aligned} \left| \int_{-1}^1 \varphi_{n_i}(x) E(x) dx \right| &\leq \|\varphi_{n_i}\|_s \cdot \|\Delta^{\frac{s}{2}} \psi E\|_{L^2(\mathbb{R})} \\ &\leq \|\varphi_{n_i}\|_s \cdot \left\{ \int_{\mathbb{R}} \left| \frac{d}{dx} (\psi E)(x) \right|^2 dx \right\}^{\frac{1}{2}}. \end{aligned}$$

But we have from (3.37) that $\sup_i \|\varphi_{n_i}\|_s < \infty$, and the second term is independent of n_i , and it can be estimated in terms of $|\zeta_1 - \zeta_2|$ by calculus. This shows that the entire functions $\{\hat{\varphi}_{n_i}(\zeta)\}$ do form an equicontinuous family. Since $\hat{\varphi}_{n_i}(\xi)$ is convergent a.e. $\xi \in \mathbb{R}$ as noted, we conclude that the entire functions $\hat{\varphi}_{n_i}(\zeta)$ converge uniformly for ζ in compact subsets of \mathbb{C} , and that the limit function is also entire analytic. But by the argument above, this limit is an extension of $\hat{f}(\xi)$, for $\xi \in \mathbb{R}$. From (3.32), we have

$$\left\langle \delta^{(n)}, Wf \right\rangle_J = (s-1)(s-2)\cdots(s-n) i^n \left(\frac{d}{d\zeta} \right)^n \hat{f}(\zeta) |_{\zeta=0}.$$

Since $f \in \mathcal{N}$, $Wf = 0$, and the left-hand side vanishes for all $n = 0, 1, 2, \dots$. Hence all the derivatives $\left(\frac{d}{d\zeta} \right)^n \hat{f}(\zeta)$ vanish at $\zeta = 0$. Since \hat{f} is analytic, it must vanish identically. Finally use (3.14) to conclude that $f = 0$ as an element of \mathcal{K}_s . This completes the proof of (c), and therefore the proof of the theorem. \square

In Section 6, we will consider more systematically the structure of systems $(\mathcal{H}_0, \mathcal{K}, J, U)$ for which $W: \mathcal{K} \rightarrow \mathcal{H}(\mathcal{K})$ is 1–1. The present construction (i.e., Theorem 3.1(c)) has the initial operator U unitary in \mathcal{H}_0 , and in fact part of a unitary one-parameter group. If the unitarity restriction on U is relaxed, then there is a richer variety of examples with $\ker(W) = \{0\}$. For example, let A denote the *unilateral shift* in $H^2 = H^2(\mathbb{T})$, and set

$$U = \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

on $\mathcal{H}_0 = H^2 \oplus H^2$. Then we show in Section 6 that the subspaces \mathcal{K} described axiomatically in Theorem 3.1 above, and which are further assumed maximal, are in 1–1 correspondence with finite positive Borel measures on $[-1, 1]$, such that $n \mapsto \int x^n d\mu(x)$ is in ℓ^2 . For those examples, the condition $\ker(W_\mu) = \{0\}$ holds if and only if $\text{supp}(\mu)$ has accumulation points in $(-1, 1)$. It holds, for example, if μ is the restriction to $[-1, 1]$ of Lebesgue measure.

4. REPRODUCING KERNELS

In the proof of part (c) of Theorem 3.1, we used the reflection J to arrive at a new Hilbert space $\mathcal{H}(\mathcal{K}_s)$. Recall that \mathcal{K}_s is the closure of $C_c(-1, 1)$ in the norm $\|\cdot\|_s$ defined as in (1.11) from the Knapp–Stein operator A_s . But in part (b) of Theorem 3.1, we showed that the system $(\mathcal{H}(\mathcal{K}_s), W, S)$ is determined uniquely from (\mathcal{K}_s, J) up to unitary equivalence. In proving part (c), we selected a particular version of $\mathcal{H}(\mathcal{K}_s)$ which turned out to contain distributions, specifically, we showed that $\left\{ \delta^{(n)} = \left(\frac{d}{dx} \right)^n \delta ; n = 0, 1, \dots \right\}$ forms an orthogonal basis in $\mathcal{H}(\mathcal{K}_s)$. Our interpretation of this is that we make the Taylor expansion around $x = 0$ into an orthogonal expansion relative to the inner product in $\mathcal{H}(\mathcal{K}_s)$. But there is an alternative construction of $\mathcal{H}(\mathcal{K}_s)$ consisting of analytic functions in

$$(4.1) \quad D := \{z \in \mathbb{C} ; |z| < 1\}.$$

This is a Hilbert space $\mathcal{H}_{\text{rep}}(s)$ constructed as a reproducing kernel Hilbert space from the kernel

$$(4.2) \quad Q_s(z, w) = (1 - z\bar{w})^{s-1}, \quad (z, w) \in D \times D.$$

It is known that there is a unique Hilbert space $\mathcal{H}_{\text{rep}}(s)$ consisting of analytic functions on D such that

$$(4.3) \quad f(w) = \langle Q_s(\cdot, w), f \rangle_{\mathcal{H}_{\text{rep}}(s)},$$

where $\langle \cdot, \cdot \rangle_{\mathcal{H}_{\text{rep}}(s)}$ is the inner product of this Hilbert space. It has the monomials $\{z^n; n = 0, 1, 2, \dots\}$ as an orthogonal basis, and we refer to [ShSh62] and [Aro50] for more details on these Hilbert spaces. It will be convenient for us to denote the kernel functions in $\mathcal{H}_{\text{rep}}(s)$,

$$(4.4) \quad q_w(z) := (1 - \bar{w}z)^{s-1}.$$

An application of (4.3) then yields

$$(4.5) \quad Q_s(w_1, w_2) = \langle q_{w_1}, q_{w_2} \rangle_{\mathcal{H}_{\text{rep}}(s)}.$$

Corollary 4.1. *The two Hilbert spaces $\mathcal{H}(\mathcal{K}_s)$ and $\mathcal{H}_{\text{rep}}(s)$, $0 < s < 1$, are naturally isomorphic with a unitary isomorphism*

$$(4.6) \quad T: \mathcal{H}(\mathcal{K}_s) \longrightarrow \mathcal{H}_{\text{rep}}(s)$$

which intertwines the respective selfadjoint scaling operators

$$(4.7) \quad (S_a f)(x) = a^{s+1} f(a^2 x)$$

and

$$(4.8) \quad (S_a^{\mathbb{C}} F)(z) = a^{s-1} F(a^{-2} z),$$

for $f \in \mathcal{H}(\mathcal{K}_s)$, $x \in \mathbb{R}$, $F \in \mathcal{H}_{\text{rep}}(s)$, $z \in D$, $a > 1$. Specifically, we have

$$(4.9) \quad TS_a = S_a^{\mathbb{C}} T.$$

Proof. While it is possible to give a direct proof along the lines of the last two pages in section 9 of [JoOl99], we will derive the result here as a direct corollary to Theorem 3.1(b), i.e., the uniqueness up to unitary equivalence. Given $a > 1$, we already established the system $(\mathcal{H}(\mathcal{K}_s), W, S_a)$ in part (c) of Theorem 3.1. We wish to show that there is a second system

$$(4.10) \quad (\mathcal{H}_{\text{rep}}(s), W^{\mathbb{C}}, S_a^{\mathbb{C}}), \quad W^{\mathbb{C}} = W_s^{\mathbb{C}},$$

which also satisfies axioms (iii)–(iv) in part (b). The s -dependence of $W = W_s$ will be suppressed in the proof for simplicity. For $S_a^{\mathbb{C}}$ we take the transformation defined in (4.8) above, and we get $W^{\mathbb{C}}: \mathcal{K}_s \rightarrow \mathcal{H}_{\text{rep}}(s)$ by the following formula:

$$(4.11) \quad (W^{\mathbb{C}} k)(z) = \int_{-1}^1 k(x) (1 - xz)^{s-1} dx$$

for $k \in \mathcal{K}_s$, and $z \in D$. To see that $S_a^{\mathbb{C}}$ in (4.8) is selfadjoint in $\mathcal{H}_{\text{rep}}(s)$, we compute the inner products as follows:

$$\begin{aligned} \langle S_a^{\mathbb{C}} q_{w_1}, q_{w_2} \rangle_{\text{rep}} &= a^{s-1} \langle q_{w_1}(a^{-2} \cdot), q_{w_2} \rangle_{\text{rep}} \\ &= a^{s-1} \langle q_{a^{-2} w_1}(\cdot), q_{w_2} \rangle_{\text{rep}} \\ &= a^{s-1} Q_s(a^{-2} w_1, w_2) \\ &= a^{s-1} (1 - a^{-2} w_1 \bar{w}_2)^{s-1} \\ &= a^{s-1} Q_s(w_1, a^{-2} w_2) \\ &= \langle q_{w_1}, S_a^{\mathbb{C}} q_{w_2} \rangle_{\text{rep}} \quad \text{for all } w_1, w_2 \in D. \end{aligned}$$

Since the kernel functions $\{q_w^{(s)}; w \in D\}$ are dense in $\mathcal{H}_{\text{rep}}(s)$ by construction, we conclude that $S_a^{\mathbb{C}}$ is indeed selfadjoint in $\mathcal{H}_{\text{rep}}(s)$ when $a > 1$ and $0 < s < 1$.

We now show that $W^{\mathbb{C}}: \mathcal{K}_s \rightarrow \mathcal{H}_{\text{rep}}(s)$ in (4.11) is contractive. For $k \in \mathcal{K}_s$, we have

$$\begin{aligned} \|W^{\mathbb{C}}k\|_{\text{rep}}^2 &= \int_{-1}^1 \int_{-1}^1 \overline{k(x)} \langle q_x, q_y \rangle_{\text{rep}} k(y) dx dy \\ &= \int_{-1}^1 \int_{-1}^1 \overline{k(x)} (1 - xy)^{s-1} k(y) dx dy \\ &= \int_{\mathbb{R}} \overline{k(x)} A_s Jk(x) dx \\ &= \langle k, Jk \rangle_{\mathcal{H}_s} \leq \|k\|_s^2, \end{aligned}$$

which shows that $W_s^{\mathbb{C}}$ is contractive as claimed. But we also proved that

$$\langle W^{\mathbb{C}}k_1, W^{\mathbb{C}}k_2 \rangle_{\text{rep}} = \langle k_1, Jk_2 \rangle_{\mathcal{H}_s}$$

for all $k_1, k_2 \in \mathcal{K}_s \subset \mathcal{H}_s$. Hence

$$(4.12) \quad (W^{\mathbb{C}})^* W^{\mathbb{C}} = P_s J P_s,$$

where P_s denotes the projection of \mathcal{H}_s onto \mathcal{K}_s . Hence axiom (iv) in the statement of Theorem 3.1(b) is also satisfied. We leave the verification of

$$(4.13) \quad S_a^{\mathbb{C}} W^{\mathbb{C}} = W^{\mathbb{C}} U P_s$$

from (b)(iii) to the reader. The conclusion of Corollary 4.1 is now immediate from Theorem 3.1(b). \square

Let $T: \mathcal{H}(\mathcal{K}_s) \rightarrow \mathcal{H}_{\text{rep}}(s)$ be the unitary isomorphism from (4.9) in the statement of Corollary 4.1. We saw in Theorem 3.1(b) that

$$TW_s = W_s^{\mathbb{C}}T.$$

Recall that $\delta^{(n)} = \left(\frac{d}{dx}\right)^n \delta$ is in $\mathcal{H}(\mathcal{K}_s)$, and we conclude that

$$T\left(\delta^{(n)}\right)(z) = (s-1)(s-2)\cdots(s-n)z^n.$$

Since T is isometric, and

$$\left\|\delta^{(n)}\right\|_{\mathcal{H}(\mathcal{K}_s)}^2 = (1-s)\cdots(n-s)n!,$$

we conclude that

$$\|z^n\|_{\mathcal{H}_{\text{rep}}(s)}^2 = \frac{n!}{(1-s)(2-s)\cdots(n-s)}.$$

We have proved the following

Corollary 4.2. *Elements of $\mathcal{H}_{\text{rep}}(s)$ may be characterized by the orthogonal expansion*

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} c_n z^n, \\ \|f\|_{\mathcal{H}_{\text{rep}}(s)}^2 &= \sum_{n=0}^{\infty} |c_n|^2 \frac{n!}{(1-s)(2-s)\cdots(n-s)}. \end{aligned}$$

5. THE HARDY SPACE $H^2(\mathbb{T})$

In this section, we return to the space $L^2(\mathbb{T})$ and its subspace $H^2(\mathbb{T})$ introduced in Section 1. Relative to the reflection $Jf(z) = f(\bar{z})$, $f \in L^2(\mathbb{T})$, we describe a family of positive subspaces defined from $H^2(\mathbb{T})$. The individual subspaces $\mathcal{K}(b)$ are positive relative to J and indexed by some function, b , say, in $H^\infty(\mathbb{T})$. However, unless $b \equiv 1$, the subspace $\mathcal{K}(b)$ is not shift invariant.

We first return to the axiomatic setup from Section 1, and we derive a formula for the contractive operator

$$(5.1) \quad W: \mathcal{K} \longrightarrow \mathcal{H}(\mathcal{K})$$

constructed from a given positive subspace $\mathcal{K} \subset \mathcal{H}_0$. Let \mathcal{H}_0 be a Hilbert space, and let J be a period-2 unitary operator in \mathcal{H}_0 . Let \mathcal{H}_\pm be the J -eigenspaces corresponding to eigenvalues ± 1 , and let P_\pm be the respective projections onto \mathcal{H}_\pm , specifically

$$(5.2) \quad P_\pm = \frac{1}{2}(I \pm J).$$

We say that a closed subspace $\mathcal{K} \subset \mathcal{H}_0$ is *positive* if

$$(5.3) \quad \langle k, Jk \rangle \geq 0 \quad \text{for all } k \in \mathcal{K}.$$

In Section 1, we proved the following:

Lemma 5.1. (a) *There is a 1–1 correspondence between the following data (i) and (ii):*
 (i) *closed positive subspaces \mathcal{K} ,*
and
 (ii) *closed subspaces $\mathcal{K}_+ \subset \mathcal{H}_+$, and contractive linear operators*

$$(5.4) \quad \Lambda: \mathcal{K}_+ \longrightarrow \mathcal{H}_-.$$

(b) *Given (i), set*

$$(5.5) \quad \mathcal{K}_+ := P_+ \mathcal{K},$$

and

$$(5.6) \quad \Lambda(P_+ k) := P_- k \quad \text{for } k \in \mathcal{K}.$$

(c) *Given (ii), set $\mathcal{K} := G(\Lambda) =$ the graph of the contraction Λ in (5.4), i.e.,*

$$(5.7) \quad \mathcal{K} = \{k_+ \oplus \Lambda k_+; k_+ \in \mathcal{K}_+\}.$$

Proof. While the details are essentially in Section 1, we sketch (i) \leftrightarrow (ii). (b) Given (i), and defining \mathcal{K}_+ and Λ by (5.5)–(5.6), we saw that \mathcal{K}_+ is *closed*, and that, by (5.3), Λ is well-defined and contractive. (c) Given (ii), the subspace \mathcal{K} in \mathcal{H}_0 , defined in (5.7), is *positive*. Indeed, if $k = k_+ + \Lambda k_+$, $k_+ \in \mathcal{K}_+$, then

$$(5.8) \quad \langle k, Jk \rangle = \|k_+\|^2 - \|\Lambda k_+\|^2 \geq 0,$$

since Λ is assumed contractive. We also easily check that \mathcal{K} in (5.7) is *closed* when (ii) holds, i.e., \mathcal{K}_+ is closed, and the operator Λ in (5.4) is contractive. \square

Corollary 5.2. *Let $\mathcal{K} \subset \mathcal{H}_0$ be a closed positive subspace as defined in Lemma 5.1 from a given J . Let $\Lambda: \mathcal{K}_+ \rightarrow \mathcal{H}_-$ be the corresponding contraction with closed domain $\mathcal{K}_+ \subset \mathcal{H}_+$, and set*

$$(5.9) \quad \mathcal{N}_+ = \{k_+ \in \mathcal{K}_+ ; \Lambda^* \Lambda k_+ = k_+\}.$$

Let

$$(5.10) \quad \mathcal{H}_+(\Lambda) = (\mathcal{K}_+/\mathcal{N}_+)^{\sim}$$

be the Hilbert space obtained by completing the quotient space $\mathcal{K}_+/\mathcal{N}_+$ relative to the Hilbert norm

$$(5.11) \quad k_+ \mapsto \left\| (I - \Lambda^* \Lambda)^{\frac{1}{2}} k_+ \right\|,$$

and let

$$(5.12) \quad W_+ : \mathcal{K}_+ \longrightarrow \mathcal{K}_+/\mathcal{N}_+ \longrightarrow \mathcal{H}_+(\Lambda)$$

be the natural contractive mapping. Then

$$(5.13) \quad W_+ = P_{\mathcal{K}} P_+ (I - \Lambda^* \Lambda)^{\frac{1}{2}} P_+ P_{\mathcal{K}},$$

where $P_{\mathcal{K}}$ denotes the projection of \mathcal{H}_0 onto \mathcal{K} , and P_{\pm} are given by (5.2). Finally there is a unitary isomorphism

$$T : \mathcal{H}_+(\Lambda) \longrightarrow \mathcal{H}(\mathcal{K})$$

which is determined by the formula

$$(5.14) \quad W = T W_+ P_+ P_{\mathcal{K}}.$$

Proof. Let \mathcal{K} be a positive subspace, and let Λ be the corresponding contraction with closed domain \mathcal{K}_+ , see Lemma 5.1. We saw that then $\mathcal{K} = G(\Lambda)$; and, if

$$(5.15) \quad k = k_+ + \Lambda k_+, \quad k_+ \in \mathcal{K}_+,$$

then

$$(5.16) \quad \langle k, Jk \rangle = \|k_+\|^2 - \|\Lambda k_+\|^2 = \langle k_+, k_+ - \Lambda^* \Lambda k_+ \rangle = \left\| (I - \Lambda^* \Lambda)^{\frac{1}{2}} k_+ \right\|^2.$$

It follows that the assignment $k_+ \mapsto k$ then passes to respective quotients

$$\mathcal{K}_+/\mathcal{N}_+ \longrightarrow \mathcal{K}/\mathcal{N},$$

where \mathcal{N}_+ is defined in (5.9). If T_0 is the corresponding operator $\mathcal{K}_+/\mathcal{N}_+ \rightarrow \mathcal{K}/\mathcal{N}$ induced by $k_+ \mapsto k_+ + \Lambda k_+$, then T_0 is isometric relative to the two new norms, and it passes to the respective completions

$$T = \tilde{T}_0 : (\mathcal{K}_+/\mathcal{N}_+)^{\sim} \longrightarrow (\mathcal{K}/\mathcal{N})^{\sim}.$$

$$\begin{array}{ccc} \parallel & & \parallel \\ \mathcal{H}_+(\Lambda) & & \mathcal{H}(\mathcal{K}) \end{array}$$

From (5.15)–(5.16), we read off formula (5.13) for the contraction $W_+ : \mathcal{K}_+ \rightarrow \mathcal{H}_+(\Lambda)$. Using again (5.16), we conclude that T satisfies (5.14). Conversely, if W and W_+ are constructed from \mathcal{K} and Λ , respectively, then, if we set $T W_+ k_+ = W k$, $k \in \mathcal{K}$, then T is isometric, and extends naturally to a unitary isomorphism of $\mathcal{H}_+(\Lambda)$ onto $\mathcal{H}(\mathcal{K})$. \square

Remark 5.3. Recent work of Arveson [Arv98] suggests a multivariable version of the construction in Section 4 above, i.e., reproducing kernels in several variables, as a candidate for a model in multivariable operator theory. With this in view, one should generalize Corollary 5.2 above to the case of a system of commuting operators $\Lambda_i: \mathcal{K}_+ \rightarrow \mathcal{H}_-$, $i = 1, \dots, d$, such that

$$\left\| \sum_{i=1}^d \Lambda_i k_i \right\|^2 \leq \sum_{k=1}^d \|k_i\|^2$$

for all $k_1, \dots, k_d, k_i \in \mathcal{K}_+$. To make the connection to the setup (5.11) in the present Corollary 5.2, note that the condition of Arveson is equivalent to the operator estimate

$$\Lambda_1 \Lambda_1^* + \dots + \Lambda_d \Lambda_d^* \leq I,$$

and the analogue of our operator from (5.11) is then

$$\left(I - \sum_{i=1}^d \Lambda_i \Lambda_i^* \right)^{\frac{1}{2}}.$$

The following observations make connections between the reflection-symmetric operator U and the subspace \mathcal{K} .

Let \mathcal{H}_+ and \mathcal{H}_- be Hilbert spaces, set

$$(5.17) \quad \mathcal{H}_0 = \mathcal{H}_+ \oplus \mathcal{H}_-, \quad J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

and let $a: \mathcal{H}_+ \rightarrow \mathcal{H}_-$ be an arbitrary operator. Then set

$$(5.18) \quad U = U(a) = \begin{pmatrix} a^* a & a^* \\ -a & a a^* \end{pmatrix}.$$

It follows that

$$(5.19) \quad J U(a) J = U(a)^* = U(-a),$$

i.e., $U(a)$ is reflection-symmetric. Moreover, $U = U(a)$ satisfies

$$U^* U = \begin{pmatrix} a^* a + (a^* a)^2 & 0 \\ 0 & a a^* + (a a^*)^2 \end{pmatrix}.$$

Conversely, every operator $U: \mathcal{H}_0 \rightarrow \mathcal{H}_0$ which satisfies

$$(5.20) \quad J U J = U^*,$$

and

$$(5.21) \quad U^* U = \left(\begin{array}{c|c} \text{operator}_1 & 0 \\ \hline 0 & \text{operator}_2 \end{array} \right)$$

relative to the decomposition (5.17) is of the form

$$(5.22) \quad U = \begin{pmatrix} s_1 & a^* \\ -a & s_2 \end{pmatrix}$$

for some operator $a: \mathcal{H}_+ \rightarrow \mathcal{H}_-$, and for two selfadjoint operators s_1 and s_2 in the respective Hilbert spaces \mathcal{H}_+ and \mathcal{H}_- , and satisfying the intertwining relation:

$$(5.23) \quad a s_1 = s_2 a.$$

Returning to the classical example from Section 1 above, let $\mathcal{H}_0 := L^2(\mathbb{T})$, and set

$$(5.24) \quad Jf(z) := f(\bar{z}), \quad f \in L^2(\mathbb{T}), \quad z \in \mathbb{T}.$$

Proposition 5.4. *Let $H^2 = H^2(\mathbb{T})$, and $H^\infty = H^\infty(\mathbb{T})$ be the usual Hardy spaces of harmonic analysis. Let $b \in H^\infty$ be given, and suppose that $\|b\|_\infty \leq 1$. Define the subspace $\mathcal{K}(b) \subset \mathcal{H}_0 (= L^2(\mathbb{T}))$ as follows:*

$$(5.25) \quad \mathcal{K}(b) = \{(1 - b(\bar{z}))k(\bar{z}) + (1 + b(z))k(z); k \in H^2\}$$

Then $\mathcal{K}(b)$ is a maximal positive subspace of \mathcal{H}_0 relative to the given reflection operator J from (5.24). Moreover, the space $\mathcal{K}(b)$ is invariant under the shift

$$(5.26) \quad Uf(z) = zf(z), \quad f \in L^2(\mathbb{T}), \quad z \in \mathbb{T},$$

if and only if $b \equiv 1$. In that case, $\mathcal{H}(\mathcal{K})$ is one-dimensional, and $S(U) = 0$.

Proof. The proof is based on Corollary 5.2 above. Since J is given by (5.24) at the outset, the two subspaces $\mathcal{H}_\pm \subset L^2(\mathbb{T})$ are then determined from (5.2), applied to J . Let $\mathcal{K} = H^2(\mathbb{T})$, and set $\mathcal{K}_\pm := P_\pm \mathcal{K}$. Then $\mathcal{K}_\pm = \mathcal{H}_\pm$, where

$$(5.27) \quad \mathcal{K}_\pm = \{k(z) \pm k(\bar{z}); k \in H^2\}.$$

Let $b \in H^\infty$, $\|b\|_\infty \leq 1$, be given, and define $\Lambda = \Lambda_b$ by

$$(5.28) \quad \Lambda(P_+k) := P_-(bk), \quad \text{for all } k \in H^2.$$

Then it follows from $\mathcal{K}_+ = \mathcal{H}_+$ that Λ is a contractive operator with domain \mathcal{H}_+ and mapping into \mathcal{H}_- . The corresponding positive subspace, see Lemma 5.1, is that which is given by (5.25). The space $\mathcal{K}(b)$ is maximally positive. A positive subspace \mathcal{K}' satisfying $\mathcal{K}(b) \subset \mathcal{K}'$ would correspond to a contractive operator Λ' mapping \mathcal{H}_+ into \mathcal{H}_- and extending Λ , in the sense that the graph of Λ' contains that of Λ . But then $\Lambda = \Lambda'$ and therefore $\mathcal{K}(b) = \mathcal{K}'$ by the uniqueness part in Lemma 5.1. This proves that $\mathcal{K}(b)$ is maximally positive in $L^2(\mathbb{T})$.

The contractive property for the operator $\Lambda = \Lambda_b$ in (5.28) follows from the two assumptions on b , i.e., $b \in H^\infty$, and $\|b\|_\infty \leq 1$. Indeed, if $k \in H^2$, then

$$\begin{aligned} \|P_-(bk)\|_2^2 &= \left\| \frac{1}{2}(-b(\bar{z})k(\bar{z}) + b(z)k(z)) \right\|_2^2 \\ &= \frac{1}{2} \left(\|bk\|_2^2 - |b(0)k(0)|^2 \right) \\ &\leq \frac{1}{2} \|bk\|_2^2 \leq \frac{1}{2} \|b\|_\infty^2 \|k\|_2^2 \leq \frac{1}{2} \|k\|_2^2 = \|P_+k\|_2^2. \end{aligned}$$

This proves that the operator $\Lambda = \Lambda_b$ in (5.28) is indeed well-defined and contractive. We then conclude from Lemma 5.1(c) that the corresponding positive subspace $\mathcal{K}(b)$ is the graph of Λ_b . An application of (5.7) from Lemma 5.1 then finally yields (5.25) as claimed.

If it were the case that $\mathcal{K}(b) (= G(\Lambda_b))$ were invariant under the shift U of (5.26), then from Beurling's theorem, there would be a unitary function $u \in L^\infty(\mathbb{T})$ such that

$$(5.29) \quad \mathcal{K}(b) = uH^2.$$

(Recall $u \in L^\infty$ is said to be unitary if the corresponding multiplication operator M_u on L^2 is unitary.) But identity in (5.29) for some unitary $u \in L^\infty$ is possible only

if the factor $(1 - b(\bar{z}))$ in (5.25) vanishes identically on \mathbb{T} , and it follows therefore that $\mathcal{K}(b)$ can only be shift-invariant if $b \equiv 1$. In this case, $\mathcal{K}(b) = \mathcal{K} = H^2$ reduces to the special case which we studied in Section 1. In that case, the contraction Λ from (5.28) reduces to $\Lambda(P_+k) = P_-k$, and

$$\langle k, Jk \rangle = \|P_+k\|^2 - \|P_-k\|^2 = |c_0|^2 \quad \text{if } k(z) = \sum_{n=0}^{\infty} c_n z^n \in H^2.$$

Hence $\mathcal{H}(\mathcal{K})$ is one-dimensional. Since

$$Uk(z) = zk(z) = c_0z + c_1z^2 + \dots$$

has zero constant term, the selfadjoint operator $S(U)$ on $\mathcal{H}(\mathcal{K})$, induced from U , is zero, and the proof is completed. \square

Elaborating on the abstract setup in Proposition 1.2, we conclude with a family of finite-dimensional positive subspaces in $H^2 \oplus H^2$.

The simplest situation when a triple $(\mathcal{H}_0, \mathcal{K}, J)$ arises in an application is the case of the Pick–Nevanlinna interpolation problem. In that case, let

$$\mathcal{H}_0 = \ell_+^2 \oplus \ell_+^2, \quad J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

$N \in \mathbb{N}$, distinct points $z_1, \dots, z_N \in D = \{z \in \mathbb{C} ; |z| < 1\}$, and $w_1, \dots, w_N \in \mathbb{C}$, be given. The Pick–Nevanlinna theorem states that there exists a function $\varphi \in H^\infty(D)$ such that $\varphi(z_i) = w_i$ for each i , and $\|\varphi\|_\infty \leq 1$ if and only if the corresponding $N \times N$ matrix $\left(\frac{1 - \bar{w}_i w_j}{1 - \bar{z}_i z_j} \right)$ is positive semidefinite. We will now assume the latter, and relate it to the \mathcal{K} -problem. Then set

$$\mathcal{K} := \left\{ \begin{pmatrix} (\sum_i c_i z_i^n)_{n=0}^\infty \\ (\sum_i c_i w_i z_i^n)_{n=0}^\infty \end{pmatrix} ; c_1, c_2, \dots, c_N \in \mathbb{C} \right\} \subset \left(\ell_+^2 \right)^\oplus.$$

It is an N -dimensional subspace, and so closed. For general vectors $k = k(c)$, $c = (c_1, \dots, c_N)$ in \mathcal{K} , the term $\langle k, Jk \rangle = \|P_+k\|^2 - \|P_-k\|^2$ computes out as

$$\sum_n \left| \sum_i c_i z_i^n \right|^2 - \sum_n \left| \sum_i c_i w_i z_i^n \right|^2 = \sum_i \sum_j \frac{1 - \bar{w}_i w_j}{1 - \bar{z}_i z_j} \bar{c}_i c_j \geq 0,$$

assuming the Pick–Nevanlinna condition.

Since we also work with the H^2 -version of ℓ_+^2 , we note that the above positive subspace \mathcal{K} has an equivalent form in $\mathcal{H}_0 = H^2 \oplus H^2$. There we have the reproducing kernel $q_z(\zeta) = (1 - \bar{z}\zeta)^{-1}$, and \mathcal{K} then takes the form of column vectors as follows:

$$\mathcal{K} = \left\{ \begin{pmatrix} \sum_i c_i q_{z_i} \\ \sum_i c_i w_i q_{z_i} \end{pmatrix} ; c_1, c_2, \dots, c_N \in \mathbb{C} \right\}.$$

The Pick–Nevanlinna problem was stated in terms of the pair $\mathcal{K}, J = (I \oplus (-I))$, but if we use instead $J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, then it is easy to check that the corresponding condition, $\langle k, Jk \rangle \geq 0$ for $k \in \mathcal{K}$, is now equivalent to the matrix order relation, $\left(\frac{\bar{w}_i + w_j}{1 - \bar{z}_i z_j} \right) \geq 0$, i.e., equivalent to

$$\sum_{i=1}^N \sum_{j=1}^N \bar{c}_i \left(\frac{\bar{w}_i + w_j}{1 - \bar{z}_i z_j} \right) c_j \geq 0 \quad \text{for all } c_1, \dots, c_N \in \mathbb{C}.$$

This alternative is in turn equivalent to a solution to the interpolation problem $\varphi(z_i) = w_i$ for each i , and $\operatorname{Re} \varphi \geq 0$ in D for some interpolating analytic function φ . Hence both of the classical interpolation problems correspond to positivity for a pair (\mathcal{K}, J) where $\mathcal{K} \subset H^2 \oplus H^2$ is as stated, but where J changes from one problem to the other.

A nice solution to both problems is presented in the classic paper [Sar67]. (See also [FaKo94].)

6. HANKEL OPERATORS

In this section, we consider the direct sum of the unilateral shift A and its adjoint A^* , i.e., $U = A \oplus A^*$. If $J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, then $JUJ = U^*$, and we solve the problem of finding the subspaces $\mathcal{K} \subset \ell_+^2 \oplus \ell_+^2$ which satisfy the positivity (ii) of Theorem 3.1, and are invariant under U . This is analogous to (and yet very different from) the classical solution of Beurling [Hel95, chapter 6] which gives the invariant subspaces for A . Recall the invariant subspaces for A are in 1–1 correspondence with the *inner functions*, i.e., functions $\xi \in H^\infty$ such that $|\xi(e^{i\theta})| = 1$ a.e. $\theta \in [-\pi, \pi)$. For our present problem with $A \oplus A^*$, we will first reduce the analysis to considering closed invariant subspaces $\mathcal{K} \subset \ell_+^2 \oplus \ell_+^2$ which are maximally positive. This reduction follows in fact from an application of Beurling’s theorem. We then show that those invariant subspaces \mathcal{K} are in 1–1 correspondence with positive and finite Borel measures μ on $[-1, 1]$ in such a way that the corresponding induced selfadjoint operator $S_\mu(A \oplus A^*)$, acting on $\mathcal{H}(\mathcal{K})$, is unitarily equivalent to multiplication by the real variable x on $L_\mu^2([-1, 1])$, i.e., $f(x) \mapsto xf(x)$, on the L^2 space given by $\int_{-1}^1 |f(x)|^2 d\mu(x) < \infty$, and defined from a finite positive measure μ on $[-1, 1]$. We also make explicit how a subspace $\mathcal{K} = \mathcal{K}_\mu$ with the desired properties may be reconstructed from some given measure μ as specified.

We first give some Hilbert-space background: Let \mathcal{H} be a Hilbert space, and let A be a bounded operator in \mathcal{H} . Then

$$(6.1) \quad U := \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix} \quad \text{on } \mathcal{H}_0 := \mathcal{H} \oplus \mathcal{H}$$

satisfies

$$(6.2) \quad JUJ = U^*$$

relative to

$$(6.3) \quad J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

i.e., the operator J on \mathcal{H}_0 is given by $J(h \oplus k) = k \oplus h$. This observation also shows that the identity (6.2) typically does not imply any special property for the operators making up U . On the other hand, the example in Section 3 had U unitary relative to the original Hilbert space \mathcal{H}_0 .

We wish to compute the correspondence $U \mapsto S(U)$ of Theorem 3.1 in the case of (6.1) and (6.3). Given a subspace $\mathcal{K} \subset \mathcal{H}_0$ such that

$$(6.4) \quad U(\mathcal{K}) \subset \mathcal{K},$$

we will pass to the new Hilbert space

$$(6.5) \quad \mathcal{H}(\mathcal{K}) = (\mathcal{K}/\mathcal{N})^\sim,$$

where $\mathcal{N} = \{k \in \mathcal{K} ; \langle k, Jk \rangle = 0\}$. We say that \mathcal{K} is the *graph* of some operator from a domain $D(\Gamma) \subset \mathcal{H}$ into \mathcal{H} , if

$$(6.6) \quad \begin{pmatrix} 0 \\ h \end{pmatrix} \in \mathcal{K} \implies h = 0.$$

But in view of (6.3), vectors of the form $\begin{pmatrix} 0 \\ h \end{pmatrix}$ are automatically in \mathcal{N} , and so do not contribute to $\mathcal{H}(\mathcal{K})$ of (6.5). We will suppose, therefore, that the spaces \mathcal{K} of (6.4) have the form $\mathcal{K} = G(\Gamma)$. Note that the operator Γ of which \mathcal{K} is the graph need not have dense domain. The subspace \mathcal{K} is said to be *positive* if $\langle k, Jk \rangle \geq 0$ for $k \in \mathcal{K}$, and *maximally positive* if it is maximal (relative to inclusion) with respect to this property. It follows from (6.3) that the maximally positive subspaces \mathcal{K} of the form $\mathcal{K} = G(\Gamma)$ correspond to operators Γ which are *dissipative*, *closed*, and have *dense domain* in \mathcal{H} . The corresponding Cayley transform

$$(6.7) \quad \Lambda := (I - \Gamma)(I + \Gamma)^{-1}$$

is then contractive and everywhere defined on \mathcal{H} , and it corresponds to the contraction also denoted Λ from Lemma 3.2. This contraction derives from the general contractive transformation

$$(6.8) \quad P_+ k \longmapsto P_- k, \quad k \in \mathcal{K},$$

where $P_{\pm} = \frac{1}{2}(I \pm J)$. Using (6.3) we get

$$P_{\pm} \begin{pmatrix} h \\ \Gamma h \end{pmatrix} = \frac{1}{2} \begin{pmatrix} h \pm \Gamma h \\ h \pm \Gamma h \end{pmatrix} \quad \text{for } h \in D(\Gamma),$$

and so

$$\left\| P_{\pm} \begin{pmatrix} h \\ \Gamma h \end{pmatrix} \right\| = \frac{1}{\sqrt{2}} \|h \pm \Gamma h\|.$$

Since (6.8) is contractive, it follows that Λ in (6.7) is well-defined and also contractive. Let A in (6.1) be the unilateral shift. Then of course U will not even be normal. Nonetheless, the possibilities for reflection symmetry yield a richer family, and we will show here that the possibilities can even be classified, i.e., if A in (6.1) is the unilateral shift.

Let $\mathcal{H} = H^2$. We will use both of the representations $f(z) = \sum_{n=0}^{\infty} c_n z^n$, and (c_0, c_1, c_2, \dots) for elements in H^2 , i.e., the function vs. its Fourier series. Hence A takes alternately the form

$$(6.9) \quad (Af)(z) = zf(z), \quad f \in H^2, \quad z \in \mathbb{T},$$

or

$$(6.10) \quad A(c_0, c_1, c_2, \dots) = (0, c_0, c_1, c_2, \dots), \quad (c_n)_{n=0}^{\infty} \in \ell^2,$$

and A^* given by $A^*(c_0, c_1, c_2, \dots) = (c_1, c_2, c_3, \dots)$.

It is immediate that, if Γ is an operator in $\mathcal{H} = H^2$, with domain $D(\Gamma)$, and graph $G(\Gamma) = \left\{ \begin{pmatrix} h \\ \Gamma h \end{pmatrix} ; h \in D(\Gamma) \right\}$, then $\mathcal{K} := G(\Gamma)$ satisfies the positivity

$$(6.11) \quad \langle k, Jk \rangle \geq 0 \quad \text{for all } k \in \mathcal{K}$$

if and only if Γ is *dissipative*, meaning

$$(6.12) \quad \operatorname{Re} \langle h, \Gamma h \rangle \geq 0 \quad \text{for all } h \in D(\Gamma).$$

It is easy to show, see, e.g., [Phil], that if Γ is dissipative, then the closure of $G(\Gamma)$, i.e., $\overline{G(\Gamma)}$, is also the graph of a dissipative operator, denoted $\bar{\Gamma}$. (An

operator is said to be *closed* if its graph is closed.) We will consider subspaces \mathcal{K} which are invariant under $U = \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix}$. But if \mathcal{K} is invariant, then so is $\overline{\mathcal{K}}$, and we will restrict attention to closed subspaces, and corresponding closed operators.

Lemma 6.1. *Let $U = \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix}$ be built from the shift A , see (6.9), and let Γ be an operator with domain $D(\Gamma)$ in H^2 , and graph $G(\Gamma)$ in $H^2 \oplus H^2$. Then*

$$(6.13) \quad U(G(\Gamma)) \subset G(\Gamma)$$

if and only if $D(\Gamma)$ is A -invariant and

$$(6.14) \quad \Gamma A = A^* \Gamma \quad \text{on } D(\Gamma).$$

Proof. Since

$$U \begin{pmatrix} h \\ \Gamma h \end{pmatrix} = \begin{pmatrix} Ah \\ A^* \Gamma h \end{pmatrix} \quad \text{for } h \in D(\Gamma),$$

we see that (6.13) holds if and only if $\Gamma Ah = A^* \Gamma h$, which is the conclusion. \square

However, the operators Γ satisfying (6.14) are the Hankel operators. Relative to the standard basis in H^2 , such a Γ has the form

$$(6.15) \quad (\Gamma x)_n = \sum_{m=0}^{\infty} \gamma_{n+m} x_m$$

for $n = 0, 1, \dots$, where γ is some sequence, $\gamma \in \ell^2$. While the bounded Hankel operators are known, the interesting ones, for reflection positivity, will be unbounded ones. (Recall $\Gamma = \Gamma_\gamma$ is bounded in H^2 if and only if there is some $\varphi \in L^\infty(\mathbb{T})$ such that $\gamma_n = \hat{\varphi}(-n)$, $n = 0, 1, \dots$, see, e.g., [Pow82].)

While we can reduce to the case when $\mathcal{K} = G(\Gamma)$ is closed in $H^2 \oplus H^2$, the domain $D(\Gamma)$ is not closed in H^2 , but only dense.

Lemma 6.2. *Let $\Gamma = \Gamma_\gamma$ be the closed operator defined in (6.15) when it is assumed that*

$$(6.16) \quad \operatorname{Re} \gamma_n \geq 0 \quad \text{for all } n = 0, 1, 2, \dots$$

Then

$$(6.17) \quad (I + \Gamma) D(\Gamma) = H^2.$$

Proof. It follows from (6.15) that the condition (6.16) on the sequence $(\gamma_n)_{n=0}^\infty$ is equivalent to Γ_γ being dissipative. Hence, since $(\gamma_n) \in \ell^2$, the operator Γ has a dense domain $D(\Gamma)$, and the closure of Γ is well-defined. We will work with the closure, and refer to Γ as the closed operator. Notice that if Γ is defined from a sequence (γ_n) , then the adjoint operator Γ^* is defined from the sequence $(\bar{\gamma}_n)$; and so, by (6.16), both are dissipative. In particular,

$$(6.18) \quad \operatorname{Re} \langle h, \Gamma^* h \rangle \geq 0$$

for all $h \in D(\Gamma^*)$. To prove (6.17), suppose $h \perp (I + \Gamma) D(\Gamma)$. Then $h \in D(\Gamma^*)$, and $\Gamma^* h = -h$. Since then $\operatorname{Re} \langle h, \Gamma^* h \rangle = -\|h\|^2$, this contradicts (6.18), unless $h = 0$. Hence $(I + \Gamma) D(\Gamma)$ is dense in H^2 . But it is also closed since Γ is closed and dissipative. \square

Theorem 6.3. *The maximally positive subspaces $\mathcal{K} \subset H^2 \oplus H^2$ which are invariant under $U = \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix}$, A the unilateral shift, have the form*

$$(6.19) \quad \mathcal{K} = G(\Gamma_\gamma) \pmod{\mathcal{N}},$$

where the sequence $\gamma \in \ell^2$ satisfies

$$(6.20) \quad 2 \operatorname{Re} \gamma_n = \int_{\mathbb{R}} x^n d\mu(x)$$

for some positive and finite Borel measure μ on the interval $I = [-1, 1] \subset \mathbb{R}$. If \mathcal{K} comes from such a measure μ , then μ is unique, and the pair $(\mathcal{H}(\mathcal{K}), S(U))$ may be taken to be $L^2(I, d\mu)$ for the Hilbert space $\mathcal{H}(\mathcal{K}_\mu)$, and multiplication by x on $L^2(I, d\mu)$ for the induced selfadjoint operator $S_\mu(U)$.

Proof. We begin with a lemma.

Lemma 6.4. *Let $\gamma_n \in \mathbb{C}$, $n = 0, 1, 2, \dots$, be a sequence such that all the sums*

$$S_\gamma(\zeta) := \sum_n \sum_m \bar{\zeta}_n \gamma_{n+m} \zeta_m$$

satisfy $S_\gamma(\zeta) \geq 0$ for sequences (ζ_n) which are eventually zero. Let μ be a positive Borel measure on $I := [-1, 1]$ with finite moments

$$\gamma_n = \int_{-1}^1 x^n d\mu(x), \quad n = 0, 1, 2, \dots$$

Let Γ be the (possibly unbounded) Hankel operator with symbol sequence (γ_n) .

- (i) *Then the following are equivalent:*
 - (a) $\mathbf{1} \in D(\Gamma)$,
 - (b) $e_n(z) := z^n \in D(\Gamma)$ for some $n \in \{0, 1, 2, \dots\}$,
 - (c) $e_n(z) := z^n \in D(\Gamma)$ for all $n \in \{0, 1, 2, \dots\}$, and
 - (d) $(\gamma_n)_{n=0}^\infty \in \ell^2$.
- (ii) *If one, and therefore all, the conditions hold, then*

$$\lim_{n \rightarrow \infty} \|\Gamma(e_n)\| = 0.$$

- (iii) *The conditions are satisfied if*

$$(6.21) \quad \int_{-1}^1 (1 - x^2)^{-\frac{1}{2}} d\mu(x) < \infty.$$

But (6.21) is more restrictive than (a)–(d) in (i).

Proof. We view $\Gamma = \Gamma_\gamma$ as an operator on H^2 , and note that, if $z^n \in D(\Gamma)$, then

$$\Gamma(z^n) = \sum_{m=0}^{\infty} \gamma_{n+m} z^m.$$

Equivalently, setting $e_n(z) := z^n$,

$$\Gamma(e_n)(z) = \sum_m \gamma_{n+m} z^m.$$

The equivalence of conditions (a)–(d) of (i) is immediate from this. Indeed, if $e_n \in D(\Gamma)$, then $\|\Gamma(e_n)\|^2 = \sum_{k=n}^{\infty} |\gamma_k|^2$. So this decides (d); and (ii) also follows.

Hence for (iii), it is enough to show that (a) follows from (6.21). Let (c_0, c_1, \dots) be a sequence which is eventually zero. Then

$$\begin{aligned} \left| \sum_{n=0}^{\infty} \gamma_n c_n \right| &= \left| \sum_{n=0}^{\infty} \int_I x^n c_n d\mu(x) \right| \leq \int_I \sum_{n=0}^{\infty} |x^n c_n| d\mu(x) \\ &\leq \int_I \left(\sum_{n=0}^{\infty} x^{2n} \right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} |c_n|^2 \right)^{\frac{1}{2}} d\mu(x) = \|(c_n)\|_{\ell^2} \cdot \int_I (1-x^2)^{-\frac{1}{2}} d\mu(x), \end{aligned}$$

and the integral on the right is finite by assumption (6.21). It follows that the sequence (γ_n) defines a bounded linear functional on $H^2 \simeq \ell_+^2$, and so it is in ℓ_+^2 by Riesz's theorem. Equivalently, $\Gamma(e_0)(z) = \sum_{n=0}^{\infty} \gamma_n z^n$ defines an element of H^2 , and so (a) holds, and in fact Γ_γ is densely defined as an operator on H^2 . \square

We now continue with the proof of Theorem 6.3. Let \mathcal{K} be given, and assume it has the properties stated in the theorem. Then from Theorem 3.1, we know that there is a selfadjoint version $S(U)$ in a Hilbert space $\mathcal{H}(\mathcal{K})$. With the data from Theorem 3.1, we also know that the pair $(\mathcal{H}(\mathcal{K}), S(U))$ is unique up to unitary equivalence. Since the spectral radius of U in the present theorem is clearly one, we get, from Theorem 3.1(v), that $\|S(U)\| \leq 1$. Suppose for the moment that $S(U)$ is realized as multiplication by x on $L^2(\mathbb{R}, \mu)$. Then the spectrum of $S_\mu(U)$ must be contained in $I = [-1, 1]$, and so the support of μ must be contained in I .

We saw in Lemmas 6.1 and 6.2 that \mathcal{K} must have the desired form (6.19) for some dissipative operator Γ with dense domain $D(\Gamma)$ in \mathcal{H} . Since $G(\Gamma)$ is mapped into itself by $\begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix}$, we get the commutation identity (6.14). Writing out the positivity (6.11) for $k = \begin{pmatrix} h \\ \Gamma h \end{pmatrix}$, $h \in D(\Gamma)$, $h(z) = \sum_{n=0}^{\infty} c_n z^n$, we get

$$\begin{aligned} (6.22) \quad \langle k, Jk \rangle &= 2 \operatorname{Re} \langle h, \Gamma h \rangle = 2 \operatorname{Re} \left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \bar{c}_n \gamma_{n+m} c_m \right) \\ &= 2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \bar{c}_n \operatorname{Re}(\gamma_{n+m}) c_m \geq 0. \end{aligned}$$

But this means that the Hamburger moment problem is solvable for the sequence $(\operatorname{Re}(\gamma_n))_{n=0}^{\infty}$. If the solution is represented as in (6.20), then it follows that $S_\mu(U)$ is represented as multiplication by x on $L^2(\mathbb{R}, \mu)$, and we saw (using Theorem 3.1(v)) that this forces μ to be supported in the interval $I = [-1, 1]$. Since $\gamma \in \ell^2$, it is known from the theory of moments that μ is unique from Γ_γ . We include the argument for why $S_\mu(U)$ is indeed multiplication by x on $L^2(\mathbb{R}, \mu)$. Returning to (6.22), we note that $S(U)$ is determined from the identity

$$\langle k, J U k \rangle = \langle k, S(U) k \rangle_J$$

for $k = \begin{pmatrix} h \\ \Gamma h \end{pmatrix}$, $h \in D(\Gamma)$; and we have:

$$J U k = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix} \begin{pmatrix} h \\ \Gamma h \end{pmatrix} = \begin{pmatrix} A^* \Gamma h \\ A h \end{pmatrix} = \begin{pmatrix} \Gamma A h \\ A h \end{pmatrix}.$$

Consider finite sums $h_1(z) = \sum_n a_n z^n$ and $h_2(z) = \sum_n b_n z^n$, and the corresponding restrictions to $z = x \in \mathbb{R}$. Using $k_1 = \begin{pmatrix} h_1 \\ \Gamma h_1 \end{pmatrix}$ and $k_2 = \begin{pmatrix} h_2 \\ \Gamma h_2 \end{pmatrix}$, we get

$$\begin{aligned} \langle k_1, S(U) k_2 \rangle_J &= \langle h_1, \Gamma A h_2 \rangle + \langle \Gamma h_1, A h_2 \rangle = 2 \sum_n \sum_m \bar{a}_n \operatorname{Re}(\gamma_{n+m}) b_{m-1} \\ &= \sum_n \sum_m \bar{a}_n \int_{\mathbb{R}} x^{n+m} d\mu(x) b_{m-1} = \int_{\mathbb{R}} \overline{h_1(x)} x h_2(x) d\mu(x). \end{aligned}$$

This concludes the proof of existence.

Proof of uniqueness in Theorem 6.3. Let μ be a finite positive Borel measure on \mathbb{R} which is supported in $[-1, 1]$, and assume that $n \mapsto \int_{-1}^1 x^n d\mu(x)$ is in ℓ^2 . We wish to reconstruct $\mathcal{K} = G(\Gamma)$ such that Γ is a closed dissipative operator with dense domain in H^2 . Note that if Γ has been found, then

$$(6.23) \quad \left\| \begin{pmatrix} h \\ \Gamma h \end{pmatrix} \right\|_J^2 = \langle h, \Gamma h \rangle + \langle \Gamma h, h \rangle = \langle h, (\Gamma + \Gamma^*) h \rangle.$$

It follows that if $\Gamma \sim (\gamma)$ for some $\gamma \in \ell^2$, then $\left\| \begin{pmatrix} h \\ \Gamma h \end{pmatrix} \right\|_J$ and therefore the corresponding norm-completion $\mathcal{H}_J(G(\Gamma))$ only depends on the sequence $(\operatorname{Re} \gamma_n)$, i.e., from (6.23), $\Gamma + \Gamma^* \sim (2 \operatorname{Re} \gamma_n)$. Equivalently, we may assume without loss of generality that the sequence (γ_n) is real-valued. Now set

$$(6.24) \quad \gamma_n := \frac{1}{2} \int_{-1}^1 x^n d\mu(x),$$

and let Γ be the corresponding positive Hankel operator. For domain $D(\Gamma)$, take the functions $h \in H^2$ which derive from corresponding $\phi \in L_\mu^2([-1, 1])$ as

$$(6.25) \quad h(z) = \int_{-1}^1 (1 - xz)^{-1} \phi(x) d\mu(x).$$

Recall A is the unilateral shift, and therefore

$$A^{*n} \gamma = (\gamma_n, \gamma_{n+1}, \dots),$$

or, in function form,

$$(6.26) \quad (A^{*n} \gamma)(z) = \gamma_n + \gamma_{n+1} z + \gamma_{n+2} z^2 + \dots.$$

We then set

$$(6.27) \quad (\Gamma h)(z) = \sum_{n=0}^{\infty} (A^{*n} \gamma)(z) \int_{-1}^1 x^n \phi(x) d\mu(x)$$

and note that Γ is a Hankel operator, which is closed with dense domain $D(\Gamma) \subset H^2$ and given by (6.25). Moreover, $\mathcal{K} = G(\Gamma)$ has the desired properties, with

$$(6.28) \quad W_\mu \begin{pmatrix} h \\ \Gamma h \end{pmatrix} (x) = h(x) \quad \text{for } h \in D(\Gamma) \subset H^2,$$

and restricting h to $(-1, 1) \subset D$. Moreover, for $\phi \in L_\mu^2([-1, 1])$,

$$(6.29) \quad (W_\mu^* \phi)(z) = \int_{-1}^1 (1 - xz)^{-1} \phi(x) d\mu(x)$$

is the function $h(z)$ given in (6.25) above. □

Remark 6.5. (Boundedness) The conditions (a)–(d) of Lemma 6.4 are satisfied if $\gamma_n = \mathcal{O}(\frac{1}{n})$, but, of course, for many examples which are not $\mathcal{O}(\frac{1}{n})$ as well. It is known in fact that the Hankel operator Γ_γ is bounded if and only if $\gamma_n = \mathcal{O}(\frac{1}{n})$. A theorem of Widom [Wid66] shows further that boundedness of the Hankel operator Γ_γ (from $\gamma_n = \int_{-1}^1 x^n d\mu(x)$ with μ a positive Borel measure) holds if and only if μ is a Carleson measure. (A positive Borel measure μ on $I = [-1, 1]$ is said to be a Carleson measure [Car62] if and only if $\mu(I \setminus (-x, x)) = \mathcal{O}(1 - x)$ for $0 < x < 1$.) It follows in particular that condition (6.21) in the Lemma 6.4 is satisfied whenever Γ_γ is assumed bounded; and further that (6.21) is more restrictive than requiring that $(\gamma_n) \in \ell^2$ where $(\gamma_n)_{n=0}^\infty$ denotes the moment sequence of μ .

Remark 6.6. The moment problem (6.20) *with* the finite support constraint seems to have been first studied in Devinatz [Dev53, Lemma 1, p. 64].

Corollary 6.7. *Let $\mathcal{K} = G(\Gamma_\gamma)$ be a subspace of $H^2 \oplus H^2$ satisfying the conditions in Theorem 6.3. Let μ be the measure on $[-1, 1]$ given by*

$$2 \operatorname{Re} \gamma_n = \int_{-1}^1 x^n d\mu(x) \quad (\in \ell^2),$$

and let

$$W_\mu : \mathcal{K} \longrightarrow L^2([-1, 1], \mu)$$

be the contractive operator which intertwines $A \oplus A^*$ with multiplication by x on $L^2([-1, 1], \mu)$, see Theorem 3.1. Then

$$\ker(W_\mu) = \{0\}$$

if and only if $\operatorname{supp}(\mu)$ has points of accumulation in $(-1, 1)$. (So in particular, we can have $\ker(W_\mu) = \{0\}$ both for measures μ which are absolutely continuous relative to Lebesgue measure on $[-1, 1]$, as well as for singular measures.)

Proof. It follows from Theorem 6.3 that

$$(6.30) \quad \left\| W_\mu \begin{pmatrix} h \\ \Gamma h \end{pmatrix} \right\|^2 = \int_{-1}^1 |h(x)|^2 d\mu(x), \quad \text{for } h \in H^2.$$

So for some $(c_n) \in \ell_+^2$, $h(z) = \sum_{n=0}^\infty c_n z^n$, and we may view $h(x)$ as the restriction to $(-1, 1)$ of the corresponding function $h(z)$ defined and analytic in $D = \{z \in \mathbb{C} ; |z| < 1\}$. If $\operatorname{supp}(\mu)$ has accumulation points in $(-1, 1)$, and $W_\mu h = 0$, then by (6.30), h vanishes on a subset of $\operatorname{supp}(\mu)$ of full measure. This subset must also have accumulation points, and since h is analytic in D , it must vanish identically.

To prove the converse, suppose $\operatorname{supp}(\mu)$ contains only isolated points. Then μ must have the form

$$\mu = \sum_n p_n \delta_{x_n},$$

where $p_n > 0$, and $\sum p_n < \infty$ and $\sum p_n (1 - x_n^2)^{-\frac{1}{2}} < \infty$. Recall μ is finite, and supported in $[-1, 1]$. Then pick $h \in H^2$, $\|h\|_{H^2} \neq 0$, such that $h(x_n) = 0$, for example $h(z) = \left(\prod_n \frac{x_n - z}{1 - x_n z} \right) z(1 - z^2)$. Then $h \in \ker(W_\mu)$. \square

Acknowledgements. The problems addressed in the present paper grew out of earlier joint work with G. Ólafsson [JoOl98, JoOl99] as well as earlier work by the present author. We are very grateful to G. Ólafsson for the benefit of ongoing discussions. We are also very grateful to Brian Treadway for excellent manuscript production.

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